SELECTIONS GENERATING NEW TOPOLOGIES

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Abstract. Every (continuous) selection for the non-empty 2-point subsets of a space X naturally defines an interval-like topology on X. In the present paper, we demonstrate that, for a second-countable zero-dimensional space X, this topology may fail to be first-countable at some (or, even any) point of X. This settles some problems stated in [7].

1. Introduction

Let X be a topological space, and let $\mathcal{F}(X)$ be the set of all non-empty closed subsets of X. Also, let $\mathcal{D} \subset \mathcal{F}(X)$. A map $f : \mathcal{D} \to X$ is a *selection* for $\mathcal{D}$ if $f(S) \in S$ for every $S \in \mathcal{D}$. A selection $f : \mathcal{D} \to X$ is *continuous* if it is continuous with respect to the relative Vietoris topology τ_V on $\mathcal{D}$. Let us recall that τ_V is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \subset \bigcup \mathcal{V} \ \text{ and } \ S \cap V \neq \varnothing, \text{ whenever } V \in \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of X.

In the sequel, all spaces are assumed to be at least Hausdorff and infinite. In the present paper, we are interested of continuous selections for $\mathcal{D}$, when $\mathcal{D}$ is a family of finite subsets of X, so we let $\mathcal{F}_n(X) = \{S \in \mathcal{F}(X) : |S| \leq n\}$, $n \geq 1$. It should be mentioned that X is naturally homeomorphic to the space $(\mathcal{F}_1(X), \tau_V)$ by identifying each point $x \in X$ with the corresponding singleton $\{x\} \in \mathcal{F}_1(X)$. In the terminology of [10], it means that the Vietoris topology is *admissible*.

In the special case when $\mathcal{D} = \mathcal{F}_2(X)$, a selection $f : \mathcal{F}_2(X) \to X$ is usually called a *weak selection* for X. Every such selection f defines an order-like relation $\preceq_f$ on X (see, [10]) by letting that $x \preceq_f y$ iff $f(\{x, y\}) = x$. For convenience, we write that $x \prec_f y$ if $x \preceq_f y$ and $x \neq y$. Let us stress the reader's attention that the relation "$\preceq_f$" may fail to be transitive. Nevertheless, to every weak selection f for X we may associate a topology $\mathcal{T}_f$ on X generated by all open "f-intervals"

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$\{y \in X : y \prec_f x\}$ and $\{y \in X : x \prec_f y\}$, $x \in X$. Some properties of this topology were studied in [4, 7]. For instance, it has been demonstrated in [7] that $\mathcal{T}_f$ is always a regular topology on X; also, that $\mathcal{T}_f$ is the usual Euclidean topology on the rational numbers $\mathbb{Q}$, whenever f is a continuous weak selection for $\mathbb{Q}$.

The topology $\mathcal{T}_f$ is close related to the variety of continuous weak selections f on a space X. For instance, most of the known selection constructions are based on some order-like properties of X, hence, in most of these constructions, the resulting topology $\mathcal{T}_f$ is, in fact, the original topology on X. From this point of view, we are mainly interested in $\mathcal{T}_f$ as an indication for the variety of continuous weak selections f on a particular space X. We are now ready to state also the main purpose of this paper. Namely, in this paper, we show that every uncountable, non-compact, second-countable, zero-dimensional space X has a continuous weak selection f so that $\mathcal{T}_f$ is not first-countable at some point of X, see Theorem 4.1. In the same theorem, we also demonstrate that f can be constructed so that $\mathcal{T}_f$ is not first-countable at any point of X provided X has an infinite pairwise disjoint cover consisting of uncountable open sets. Several applications follow by Theorem 4.1. For instance, it implies that there exists a continuous weak selection f on the irrational numbers $\mathbb{P}$ so that $\mathcal{T}_f$ is not first-countable at any point of $\mathbb{P}$ (see, Corollary 4.2), which answers in negative to [7, Question 2], and in positive to [7, Question 3]. Another interesting consequence is that an uncountable metrizable space X, with a covering dimension $\dim(X) = 0$, is compact if and only if $\mathcal{T}_f$ is second-countable for every continuous weak selection f on X (see, Corollary 4.4). For other applications, we refer the interested reader to Sections 4 and 5 of the paper.

A preparation for the proof of Theorem 4.1 is done in the next Sections 2 and 3, while its proof will be finally accomplished in Section 4. A part of this preparation is based on a special case of a criterion for the existence of continuous weak selections (see, Theorem 5.1), which is analogous to a result of Eilenberg of orderability [1], and has a list of interesting independent consequences (see, Section 5).

In conclusion, the second author would like to express his best gratitude to Professor S. García-Ferreira for his support and hospitality, and for discussing some questions related to this research.

2. A relation generated by weak selections

Let X be a set, and let $E \subset X \times X$ be a *relation* on X. As usual, we write xEy to denote that $(x, y) \in E$. Now, let us recall that a relation E on X is *anti-symmetric* if xEy and yEx implies $x = y$. Following [7], we say that an anti-symmetric relation E on X is a *selection relation* if xEy or yEx for every $x, y \in X$. Let us stress the attention that, in this terminology, a relation E on

X is a *linear order* on X if E is a selection relation which is also transitive (i.e., xEy and yEz implies xEz).

It should be mentioned that the set of all possible weak selections for X corresponds precisely to all possible selection relations on X. Namely, any selection relation E on X defines a weak selection f_E by letting $f_E(\{x,y\}) = x$ iff xEy. On the other hand, if f is a weak selection for X, then the order-like relation $\preceq_f$ generated by f is a selection relation. In the sequel, we will refer to $\preceq_f$ as a selection relation.

In the present section, we are interested in a natural extension of such relations to partial orders on the subsets of X. Following [3], for a selection relation "$\preceq$" and subsets $B, C \subset X$, not necessarily non-empty, we shall write that $B \preceq C$ (respectively, $B \prec C$) if $y \preceq z$ (respectively, $y \prec z$) for every $y \in B$ and $z \in C$. Obviously, $B \prec C$ implies $B \cap C = \varnothing$.

In terms of this partial order, we have the following simple criterion for continuity in $\mathcal{F}_2(X)$, which is, in fact, [4, Theorem 3.1].

Proposition 2.1 ([4])**.** *Let X be a space, f be a weak selection for X, and let "$\preceq_f$" be the selection relation generated by f. Also, let $x, y \in X$ be such that $x \prec_f y$. Then, f is continuous at $\{x,y\}$ if and only if there are open sets U and V such that $x \in U$, $y \in V$, and $U \prec_f V$.*

On the other hand, let us mention the following property of weak selections, it appeared in an explicit or implicit way in several papers, and was summarized in [6, Proposition 4.1].

Proposition 2.2. *Let X be a space, and let f be a weak selection. Then, f is continuous on the singletons of X.*

Motivated by Propositions 2.1 and 2.2, we may consider only the subset

$$[X]^2 = \mathcal{F}_2(X) \setminus \mathcal{F}_1(X) = \{S \in \mathcal{F}_2(X) : |S| = 2\},$$

which will play a crucial role in this paper. In fact, we will make no difference between weak selections $f : \mathcal{F}_2(X) \to X$ and weak selections $f : [X]^2 \to X$.

The following simple observation about special weak selections will be useful in our further considerations.

Proposition 2.3. *Let X be a space which has a continuous weak selection, and an infinite pairwise disjoint cover $\mathcal{V}$ consisting of non-empty open subsets. Then, there exists a continuous weak selection $g : [X]^2 \to X$ such that $\mathcal{V}$ is an unbounded well-ordered set with respect to the selection relation generated by g.*

Proof. Let $f : [X]^2 \to X$ be a continuous weak selection. Also, let $h : \delta \to \mathcal{V}$ be a one-to-one map, where $\delta = |\mathcal{V}|$. Then, for every $x \in X$, let $\alpha(x) < \delta$ be so that

$x \in h(\alpha(x))$. Finally, define $g : [X]^2 \to X$ by letting for distinct points $x, y \in X$ that $g(\{x, y\}) = x$ if $\alpha(x) < \alpha(y)$, and $g(\{x, y\}) = f(\{x, y\})$ if $\alpha(x) = \alpha(y)$. Clearly, g is continuous because so is f, while $\mathcal{V}$ is a discrete open cover of X. On the other hand, by the definition of g, the selection relation $\preceq_g$ defines the same order on $\mathcal{V}$ as that one of the infinite cardinal δ, hence $\mathcal{V}$ is unbounded and well-ordered with respect to $\preceq_g$. □

We conclude this section with some properties of the topology generated by weak selections. Suppose that f is a weak selection for X, and $\preceq_f$ is the selection relation generated by f. For every $x \in X$, we will consider the corresponding open "f-intervals"

$$\mathbb{I}_f(x, \infty) = \{y \in X : x \prec_f y\}, \quad \text{and} \quad \mathbb{I}_f(\infty, x) = \{y \in X : y \prec_f x\}.$$

Also, for convenience, we let

$$\mathcal{I}_f(X) = \{\mathbb{I}_f(\infty, x), \mathbb{I}_f(x, \infty) : x \in X\}.$$

In these terms, the topology $\mathcal{T}_f$ is generated by all finite intersections of members of $\mathcal{I}_f(X)$. This is the place to stress the reader's attention that, in general, the relation $\preceq_f$ is not transitive. Hence, we may have points $x, y, z \in X$ which generate an infinite "monotone" sequence

$$\cdots \prec_f x \prec_f y \prec_f z \prec_f x \prec_f \cdots$$

and, as we will see in the next sections, this can lead to situations when

$$\{t \in X : x \prec_f t \prec_f y\} \neq \varnothing \neq \{t \in X : y \prec_f t \prec_f x\}.$$

Motivated by this, for every two points $a, b \in X$ we will associate the set

$$\mathbb{I}_f(a, b) = \mathbb{I}_f(a, \infty) \cap \mathbb{I}_f(\infty, b) = \{x \in X : a \prec_f x \prec_f b\}.$$

However, we don't require that $a \prec_f b$, hence both f-intervals $\mathbb{I}_f(a, b)$ and $\mathbb{I}_f(b, a)$ make sense, and could be non-empty.

In what follows, we shall say that a point $x \in X$ is an *f-cutting point* if there are points $a, b \in X$, with $x \in \mathbb{I}_f(a, b)$. Otherwise, we shall say that x is an *f-extreme point* of X. Clearly, X may have at most two f-extreme points, which could be different for different selections f.

Proposition 2.4. *Let X be a space, f be a weak selection for X, and let $A, B \subset X$ be non-empty subsets such that*

$$\mathbb{I}_f(A, B) = \bigcap \{\mathbb{I}_f(a, b) : (a, b) \in A \times B\} \neq \varnothing.$$

Then, $A \cap B = \varnothing$. In particular, if $x \in X$ is an f-cutting point and $U \in \mathcal{T}_f$, then $x \in U$ if and only if there are non-empty finite disjoint subsets $A, B \subset X$, with $x \in \mathbb{I}_f(A, B) \subset U$.

Proof. The first part of this statement follows from the fact that $\mathbb{I}_f(z,z) = \varnothing$ for every $z \in X$. As for the second part, by the definition of $\mathcal{T}_f$, $x \in U$ if and only if there is a finite set $\mathcal{K} \subset \mathcal{I}_f(X)$, with $x \in \bigcap \mathcal{K} \subset U$. On the other hand, there should be points $a, b \in X$, with $x \in \mathbb{I}_f(a,b)$, because x is f-cutting. Let $A_0 = \{y \in X : \mathbb{I}_f(y,\infty) \in \mathcal{K}\}$ and $B_0 = \{z \in X : \mathbb{I}_f(\infty, z) \in \mathcal{K}\}$. Then, $A = A_0 \cup \{a\}$ and $B = B_0 \cup \{b\}$ are as required. □

3. A condition for continuity of weak selections

Lemma 3.1. *Let X be a space, $f : [X]^2 \to X$ be a selection, and let $\preceq_f$ be the selection relation generated by f. Then, f is continuous if and only if the set $\mathcal{L} = \{(x,y) \in X^2 : x \prec_f y\}$ is open in X^2. In particular, if f is continuous, then the map $h : \mathcal{L} \to [X]^2$, defined by $h((x,y)) = \{x,y\}$, $(x,y) \in \mathcal{L}$, is a homeomorphism.*

Proof. Take distinct points $x, y \in X$ so that $(x,y) \in \mathcal{L}$, i.e. $x \prec_f y$. Then, by Proposition 2.1, f is continuous at $\{x,y\}$ if and only if there are open sets $U, V \subset X$ such that $x \in U$, $y \in V$, and $U \prec_f V$. According to the definition of $\mathcal{L}$, this implies that f is continuous at $\{x,y\}$ if and only if there are disjoint open subsets $U, V \subset X$ such that $(x,y) \in U \times V \subset \mathcal{L}$. In particular, if f is continuous, then the map h is a continuous open bijection, which completes the proof. □

Lemma 3.1 is related to a natural construction of continuous weak selections. Towards this end, if $Z \subset X^2$, let us agree to say that $\pi : Z \to X$ is a *projection* if $\pi((x,y)) \in \{x,y\}$ for every $(x,y) \in Z$. It should be mentioned that, for any subset $Z \subset X^2$, we have always two standard continuous projections $\pi_i : Z \to X$, $i = 0, 1$, which are defined by $\pi_0((x,y)) = x$ and $\pi_1((x,y)) = y$, $(x,y) \in Z$. Here is another example of continuous projections, which will play an important role in the next section.

Example 3.2. Let X be a space, $Z \subset X^2$, $\mathcal{U}$ be a discrete open cover of Z, and let $\xi : \mathcal{U} \to 2 = \{0,1\}$ be an arbitrary map. Define a map $\pi : Z \to X$ by letting for $(x,y) \in Z$ that $\pi((x,y)) = \pi_{\xi(U)}((x,y))$ if $(x,y) \in U \in \mathcal{U}$. Then, π is a continuous projection such that $\pi \restriction U = \pi_{\xi(U)} \restriction U$, $U \in \mathcal{U}$.

Proof. Follows from the fact that $\mathcal{U}$ is a discrete open cover of Z. □

According to Lemma 3.1, we now have the following immediate consequence, which provides a possible way to construction continuous weak selections from given ones.

Corollary 3.3. *Let X be a space, $f : [X]^2 \to X$ be a continuous selection, and let $\mathcal{L}$ and $h : \mathcal{L} \to [X]^2$ be as in Lemma 3.1. Also, let $\pi : \mathcal{L} \to X$ be a continuous projection. Then, $g = \pi \circ h^{-1} : [X]^2 \to X$ is a continuous selection.*

4. A construction of continuous weak selections

Theorem 4.1. *Let X be an uncountable, non-compact, second-countable, zero-dimensional space. Then, X has a continuous weak selection f such that $\mathcal{T}_f$ is not first-countable at some point of X. If, moreover, X has an infinite pairwise disjoint open cover consisting of uncountable sets, then X has a continuous weak selection f such that $\mathcal{T}_f$ is not first-countable at any point of X.*

Proof. First of all, let us observe that X is regular because it has a base of clopen sets. Hence, by the Urysohn's metrization theorem [12] (see, also, [2]), X is metrizable. Also, $\dim(X) = 0$ [13] (see, also, [2]) because X is a Lindelöf space being regular and second-countable.

In case X has an infinite cover consisting of pairwise disjoint uncountable open sets, we let this cover to be $\mathcal{V}$. Otherwise, let us observe that X has an infinite cover $\mathcal{V}$ consisting of non-empty pairwise disjoint open sets such that at least three members of $\mathcal{V}$ are uncountable. To this end, let Z be the set of all points $x \in X$ such that x has a local base consisting of uncountable open sets. Then, Z must be closed because every neighbourhood of a point $z \in \overline{Z}$ will contain a point of Z. In this case, $X \setminus Z$ must be countable. Namely, take a countable base $\mathcal{O}$ for the topology of X, and then observe that $X \setminus Z = \bigcup\{O \in \mathcal{O} : |O| \leq \omega\}$. Since X is uncountable, Z must be also uncountable. Therefore, each open $V \subset X$, with $V \cap Z \neq \varnothing$, must be uncountable. Thus, using that $\dim(X) = 0$, we can take $\mathcal{V}$ to be an infinite cover of X consisting of non-empty pairwise disjoint open sets, with $|\{V \in \mathcal{V} : V \cap Z \neq \varnothing\}| \geq 3$.

Having already constructed the cover $\mathcal{V}$, let $\mathcal{B}$ be a countable base for the topology of X consisting of non-empty clopen subsets such that $\mathcal{B}$ is a refinement of $\mathcal{V}$. In what follows, we will use $\mathcal{D}(\mathcal{B})$ to denote the set of all non-empty subsets $\mathcal{W} \subset \mathcal{B}$ which are finite and pairwise disjoint. Next, for every $\mathcal{W} \in \mathcal{D}(\mathcal{B})$, let $2^{\mathcal{W}}$ be the set of all maps $\mu : \mathcal{W} \to 2 = \{0, 1\}$. Also, we let

$$\mathcal{M}_0 = \bigcup \left\{2^{\mathcal{W}} : \mathcal{W} \in \mathcal{D}(\mathcal{B})\right\}.$$

Finally, for every $\mu \in \mathcal{M}_0$, we let $\mathrm{Dom}(\mu)$ to be the domain of μ, which is clearly a non-empty finite and pairwise disjoint subset of $\mathcal{B}$.

Note that X has a continuous weak selection, because it is a subset of the Cantor set [13] (see, also, [2]) being a regular space with a countable clopen base. Hence, by Proposition 2.3, X has a continuous weak selection g such that $\mathcal{V}$ is an unbounded well-ordered set with respect to the selection relation "$\preceq_g$" generated by g. Now, let $\mathcal{V}^* = \{V \in \mathcal{V} : |V| > \omega\}$, which, by construction, has the property that $|\mathcal{V}^*| \geq 3$. Next, let $V^* = \min_{\preceq_g} \mathcal{V}^*$, $x^* \in V^*$, and let $W^* \in \mathcal{B}$ be such that $x^* \in W^* \subset V^*$. Finally, define

$$\mu_i^* : \{W^*\} \to 2,\ i = 0, 1, \quad \text{by } \mu_i^*(W^*) = 1 - i. \tag{4.1}$$

Thus, we get two different elements $\mu_0^*, \mu_1^* \in \mathcal{M}_0$, so we let

$$\alpha(\mu_0^*) = \min_{\preceq_g} \mathcal{V}^* \setminus \{V^*\} \quad \text{and} \quad \alpha(\mu_1^*) = \min_{\preceq_g} \mathcal{V}^* \setminus \{V^* \cup \alpha(\mu_0^*)\}.$$

For later use, let us observe that

$$W^* \prec_g \alpha(\mu_0^*) \prec_g \alpha(\mu_1^*), \quad \text{while both } \alpha(\mu_0^*) \text{ and } \alpha(\mu_1^*) \text{ are uncountable.} \tag{4.2}$$

Now, we are going to extend the map $\alpha : \{\mu_0^*, \mu_1^*\} \to \mathcal{V}$ to an injective map $\alpha : \mathcal{M}_0 \to \mathcal{V}$ such that, for every $\mu \in \mathcal{M}_0$,

$$W \prec_g \alpha(\mu), \quad \text{whenever } W \in \mathrm{Dom}(\mu). \tag{4.3}$$

This can be done by transfinite induction because $\mathcal{M}_0$ is countable, while $\mathcal{V}$ is infinite, hence $|\mathcal{M}_0| \leq |\mathcal{V}|$. Namely, take a well-ordering $\ll$ on $\mathcal{M}_0$ as that of the first infinite ordinal ω such that $\mu_0^* = \min_{\ll} \mathcal{M}_0$ and $\mu_1^* = \min_{\ll} \mathcal{M}_0 \setminus \{\mu_0^*\}$. Next, suppose that $\alpha(\nu)$ has been already defined for every $\nu \ll \mu$ and some $\mu \in \mathcal{M}_0$, with $\mu \gg \mu_1^*$. Then,

$$\mathcal{V}_\mu = \{\alpha(\nu) : \nu \ll \mu\} \cup \{V \in \mathcal{V} : W \subset V \text{ for some } W \in \mathrm{Dom}(\mu)\},$$

is a non-empty finite subset of $\mathcal{V}$, while $\mathcal{V}$ is unbounded. Hence the set

$$\mathcal{V}^\mu = \{V \in \mathcal{V} : \max_{\preceq_g} \mathcal{V}_\mu \prec_g V\}$$

is also non-empty. So, we may define $\alpha(\mu) = \min_{\preceq_g} \mathcal{V}^\mu$, which completes the construction.

Now, for convenience, we let $\vartheta : \{X\} \to 2 = \{0,1\}$ to be the map $\vartheta(X) = 0$, and $\mathcal{M} = \mathcal{M}_0 \cup \{\vartheta\}$. Also, we let $A_0 = \bigcup(\mathcal{V} \setminus \alpha(\mathcal{M}_0))$, $\alpha(\vartheta) = A_0$, and $\mathcal{A} = \alpha(\mathcal{M}_0) \cup \{A_0\}$. Thus, we get a discrete partition $\mathcal{A}$ of X and a one-to-one map $\alpha : \mathcal{M} \to \mathcal{A}$ such that (4.3) holds for every $\mu \in \mathcal{M}_0$. Keeping in mind this, we are going to construct a discrete open partition $\mathcal{U} = \left\{U^i_{\alpha(\mu)} : \mu \in \mathcal{M} \text{ and } i = 0,1\right\}$ of the set $\mathcal{L} = \{(x,y) \in X^2 : x \prec_g y\}$. Towards this end, to every subset $F \subset X$ and every $k \in 2$, we are going to associate another subset $\mathcal{S}(k,F)$ which is defined by

$$\mathcal{S}(k,F) = \begin{cases} X \setminus F & \text{if } k = 0 \\ F & \text{if } k = 1. \end{cases} \tag{4.4}$$

Then, for every $\mu \in \mathcal{M}$, we define

$$U^0_{\alpha(\mu)} = \mathcal{L} \cap \left(\bigcup \{\mathcal{S}(1-\mu(W), W) \times \alpha(\mu) : W \in \mathrm{Dom}(\mu)\}\right), \tag{4.5}$$

and

$$U^1_{\alpha(\mu)} = \mathcal{L} \cap \left(\bigcup \{\mathcal{S}(\mu(W), W) \times \alpha(\mu) : W \in \mathrm{Dom}(\mu)\}\right). \tag{4.6}$$

It is easy to observe that $U^0_{\alpha(\mu)} \cup U^1_{\alpha(\mu)} = \mathcal{L} \cap (X \times \alpha(\mu))$, hence $\mathcal{U}$ is a partition of $\mathcal{L}$. Also, $\mathcal{U}$ is defined only by products of clopen sets, hence it is clopen as well.

Finally, we define a map $\xi : \mathcal{U} \to 2$ by letting $\xi(U) = i$ if $U = U^i_{\alpha(\mu)}$ for some $\mu \in \mathcal{M}$. Thus, by Example 3.2, we get a continuous projection $\pi : \mathcal{L} \to X$, with $\pi \restriction U^i_{\alpha(\mu)} = \pi_i \restriction U^i_{\alpha(\mu)}$, $\mu \in \mathcal{M}$ and $i = 0, 1$. Hence, by Corollary 3.3, $f = \pi \circ h^{-1}$ is a continuous selection for $[X]^2$. In what follows, let "$\preceq_f$" be the selection relation generated by f.

We are going to show that f is as required, which is based on a crucial property of f. To prepare for this, take $\mu \in \mathcal{M}_0$, $W \in \mathrm{Dom}(\mu)$ and $(x, y) \in W \times \alpha(\mu)$, and let us observe that

$$f(\{x, y\}) = \begin{cases} x & \text{if } \mu(W) = 0, \\ y & \text{if } \mu(W) = 1. \end{cases} \tag{4.7}$$

Indeed, by (4.3), we have $W \times \alpha(\mu) \subset \mathcal{L}$. If $\mu(W) = 0$, then, by (4.4) and (4.5),

$$\mathcal{S}(1 - \mu(W), W) \times \alpha(\mu) = \mathcal{S}(1, W) \times \alpha(\mu) = W \times \alpha(\mu) \subset U^0_{\alpha(\mu)},$$

so $(x, y) \in U^0_{\alpha(\mu)}$, which implies that $f(\{x, y\}) = \pi_0((x, y)) = x$. If $\mu(W) = 1$, then, in the same way, by (4.4) and (4.6),

$$\mathcal{S}(\mu(W), W) \times \alpha(\mu) = \mathcal{S}(1, W) \times \alpha(\mu) = W \times \alpha(\mu) \subset U^1_{\alpha(\mu)},$$

and therefore $f(\{x, y\}) = \pi_1((x, y)) = y$ which completes the verification of (4.7).

We are now ready for the promised property of f. Namely, take $y, z \in X$ and non-empty finite disjoint subsets $A, B \subset X$. Then,

$$\{y, z\} \cap (A \cup B) = \varnothing \quad \text{implies} \quad \mathbb{I}_f(A, B) \setminus \mathbb{I}_f(y, z) \neq \varnothing. \tag{4.8}$$

Indeed, consider the finite set $K = A \cup B \cup \{y, z\}$, and then take a pairwise disjoint family $\mathcal{W} = \{W_x : x \in K\} \subset \mathcal{B}$ such that $x \in W_x$ for every $x \in K$. Next, define $\mu : \mathcal{W} \to 2$ by letting for $x \in K$ that $\mu(W_x) = 0$ if $x \in A$ or $x = z$, and $\mu(W_x) = 1$ otherwise. Thus, we get a particular element μ of $\mathcal{M}_0$. Take a point $e \in \alpha(\mu)$, and let us observe that, by the definition of μ and (4.7), we now have that $x \prec_f e$ if $x \in A$ or $x = z$, while $e \prec_f x$ if $x \in B$ or $x = y$. That is, $e \in \mathbb{I}_f(A, B)$, but $e \notin \mathbb{I}_f(y, z)$.

We are finally ready to show that the selection f is as required. Towards this end, let us show that each point of X is an f-cutting point. Namely, take a point $x \in X$, and let $W \in \mathcal{B}$ be such that $x \in W$. Just like in (4.1), define $\mu_i : \{W\} \to 2$, $i = 0, 1$, by $\mu_i(W) = 1 - i$. Thus, we get two different elements $\mu_0, \mu_1 \in \mathcal{M}_0$, so $\alpha(\mu_0) \neq \alpha(\mu_1)$. Then,

$$x \in \mathbb{I}_f(y_0, y_1), \quad \text{whenever } (y_0, y_1) \in \alpha(\mu_0) \times \alpha(\mu_1). \tag{4.9}$$

Indeed, by (4.7), $\mu_0(W) = 1$ implies $f(\{x, y_0\}) = y_0$ because $(x, y_0) \in W \times \alpha(\mu_0)$, while $\mu_1(W) = 0$ implies $f(\{x, y_1\}) = x$ because $(x, y_1) \in W \times \alpha(\mu_1)$.

In fact, (4.9) implies that $\mathcal{T}_f$ is not first countable at any point $x \in X$ for which both $\alpha(\mu_0)$ and $\alpha(\mu_1)$ are uncountable. Namely, if we assume that $\mathcal{T}_f$ is first-countable at $x \in X$, then, by Proposition 2.4, there should be a countable set $E(x) \subset X$ such that for every neighbourhood $U \in \mathcal{T}_f$ of x there are non-empty finite disjoint subsets $A, B \subset E(x)$, with $x \in \mathbb{I}_f(A, B) \subset U$. On the other hand, there are points $y_i \in \alpha(\mu_i) \setminus E(x)$, $i = 0, 1$, because both $\alpha(\mu_0)$ and $\alpha(\mu_1)$ are uncountable. However, by (4.9), this implies that $x \in \mathbb{I}_f(y_0, y_1)$, while, by (4.8), it implies that $\mathbb{I}_f(A, B) \setminus \mathbb{I}_f(y_0, y_1) \neq \varnothing$ for every two non-empty finite disjoint subsets $A, B \subset E(x)$. The contradiction so obtained implies that $\mathcal{T}_f$ is not first-countable at x. In particular, by (4.2), it now implies that $\mathcal{T}_f$ is not first-countable at the point x^* selected at the beginning of this proof. Also, if each element of $\mathcal{V}$ is uncountable, then $\mathcal{T}_f$ will be not first-countable at any point of X, which completes the proof. ☐

The following is an immediate consequence of Theorem 4.1 which answers in negative to [7, Question 2], and in positive to [7, Question 3].

Corollary 4.2. *Let $\mathbb{P}$ be the set of the irrational numbers endowed with the usual Euclidean topology. Then, $\mathbb{P}$ has a continuous weak selection f so that $\mathcal{T}_f$ is not first-countable at any point of $\mathbb{P}$.*

Here is another interesting consequence of Theorem 4.1.

Corollary 4.3. *Let $\mathfrak{C}$ be the Cantor set, and let $p \in \mathfrak{C}$. Then, $X = \mathfrak{C} \setminus \{p\}$ has a continuous weak selection f so that $\mathcal{T}_f$ is not first-countable at any point of X. In particular, f cannot be extended to a continuous weak selection for $\mathfrak{C}$.*

Concerning the statements of Corollary 4.3, let us explicitly mention that if $(X, \mathcal{T})$ is a compact space, then $\mathcal{T}_f = \mathcal{T}$ for every continuous weak selection f for X, see [4, Proposition 4.1]. The same is true if $(X, \mathcal{T})$ is connected and locally connected, see [11, Lemma 11.3], but we don't know if this holds outside the both classes of spaces mentioned above, see [7, Question 4]. Related to this, we have the following further consequence of Theorem 4.1.

Corollary 4.4. *Let X be an uncountable metrizable space X, with $\dim(X) = 0$. Then, X is compact if and only if $\mathcal{T}_f$ is second-countable for every continuous weak selection f for X.*

Proof. If X is compact and $\mathcal{T}$ is the topology of X, then $\mathcal{T}_f = \mathcal{T}$ for every continuous weak selection f for X. Note that, in this case, X has at least one continuous weak selection because it is a subset of the Cantor set. Suppose now that X is not compact. We have the following two possibilities. If X is not separable, then it should have an uncountable discrete cover consisting of open sets because $\dim(X) = 0$. On the other hand, it has a continuous weak selection because, for instance, the topology of X is generated by some linear order on X, [8, 9] (see, also, [2]). Then, by Proposition 2.3, X has a continuous weak selection g so that $\mathcal{T}_g$ is not second-countable. In case X is separable, by Theorem 4.1, we get again that X has a continuous weak selection f so that $\mathcal{T}_f$ is not second-countable. Thus, if $\mathcal{T}_f$ is second-countable for any continuous weak selection for X, then X must be compact. □

5. A condition for the existence of continuous weak selections

In this section, we demonstrate some natural relations between the existence of continuous weak selections, and the Eilenberg's result [1, Theorem I] about ordered topological spaces. In fact, these relations were naturally incorporated in Lemma 3.1. Now, we provide another reading of this lemma in terms of Eilenberg's condition of orderability in [1], and some possible further consequences. Namely, following the terminology in [1], for a space X, we let $\Delta(X) = \{(x, x) : x \in X\}$, and $P(X) = X^2 \setminus \Delta(X)$. Also, we consider the map $\Lambda : P(X) \to P(X)$ defined by $\Lambda(x, y) = (y, x)$, whenever $(x, y) \in P(X)$.

Theorem 5.1. *A space X has a continuous weak selection if and only if there are non-empty open subsets $\mathcal{L}$ and $\mathcal{R}$ of $P(X)$ such that $\mathcal{L} \cap \mathcal{R} = \varnothing$, $\mathcal{L} \cup \mathcal{R} = P(X)$, and $\mathcal{R} = \Lambda(\mathcal{L})$.*

Proof. Let f be a continuous weak selection for X. Following Lemma 3.1, we let $\mathcal{L} = \{(x, y) \in P(X) : x \prec_f y\}$ and $\mathcal{R} = \Lambda(\mathcal{L})$, where $\preceq_f$ is the selection relation generated by f. Then, $\mathcal{L} \cap \mathcal{R} = \varnothing$ and $\mathcal{L} \cup \mathcal{R} = P(X)$. Also, $\mathcal{L}$ is open in $P(X)$ if and only if $\mathcal{R} \subset P(X)$ is open. Hence, Lemma 3.1 completes the proof of this implication.

Suppose now that there are open subsets $\mathcal{L}, \mathcal{R} \subset P(X)$ such that $\mathcal{L} \cap \mathcal{R} = \varnothing$, $\mathcal{L} \cup \mathcal{R} = P(X)$, and $\mathcal{R} = \Lambda(\mathcal{L})$. Next, define a selection $f : [X]^2 \to X$ by letting $f(\{x, y\}) = x$ if $(x, y) \in \mathcal{L}$. Note that if $x \neq y$, then either $(x, y) \in \mathcal{L}$ or $(y, x) \in \mathcal{L}$, so our definition is correct. That f is continuous, it follows by Lemma 3.1. □

Corollary 5.2. *Let X be a space which has a continuous weak selection. Then, $P(X)$ is not connected.*

Proof. According to Theorem 5.1, there are non-empty open subsets $\mathcal{L}, \mathcal{R} \subset P(X)$ such that $\mathcal{L} \cap \mathcal{R} = \varnothing$ and $\mathcal{L} \cup \mathcal{R} = P(X)$. In particular, both $\mathcal{L}$ and $\mathcal{R}$ are clopen in $P(X)$. □

Corollary 5.3. *Let X be a connected space which has a continuous weak selection. Then, $P(X)$ consists of two components $\mathcal{L}$ and $\mathcal{R}$ such that $\mathcal{R} = \Lambda(\mathcal{L})$.*

Proof. By Corollary 5.2, $P(X)$ is not connected. The rest of the proof follows precisely that one in [1, (3.1)]. □

Corollary 5.4. *Let X be a connected space which has a continuous weak selection, and let $E(X) = \{x \in X : X \setminus \{x\}$ is connected$\}$. Then, $|E(X)| \leq 2$.*

Proof. By Corollary 5.3, $P(X)$ has exactly two components $\mathcal{L}$ and $\mathcal{R}$ such that $\mathcal{R} = \Lambda(\mathcal{L})$. Suppose that $x, y \in E(X)$ are distinct points, with $(X \setminus \{x\}) \times \{x\} \subset \mathcal{L}$ and $(X \setminus \{y\}) \times \{y\}) \subset \mathcal{L}$. Then, we get that $(y, x) \in \mathcal{L}$ and $(x, y) \in \mathcal{L}$, which is impossible because $\Lambda(x, y) = (y, x)$. Thus, we get that there exists at most one point $x \in E(X)$, with $(X \setminus \{x\}) \times \{x\} \subset \mathcal{L}$. In the same way, there exists at most one point $y \in E(X)$, with $(X \setminus \{y\}) \times \{y\} \subset \mathcal{R}$. Hence, $|E(X)| \leq 2$. □

Now we get the following result, which, for instance, implies the well-known fact that the unit circle has no continuous selection. Let us recall that we consider only infinite spaces.

Corollary 5.5. *Let X be a connected space which has a continuous weak selection. Then, there exists a point $x \in X$ such that $X \setminus \{x\}$ is not connected.*

We complete this list of consequences with the following one related to continuous weak selections on product spaces.

Corollary 5.6. *Let X be a space such that X^2 has a continuous weak selection. Then, X must be totally disconnected.*

Proof. Let Z be a connected component of X, and let us show that $|Z| = 1$. So, suppose that this fails, hence $|Z| \geq 2$. On the other hand, Z^2 has a continuous weak selection because $Z^2 \subset X^2$. Hence, by Corollary 5.5, there should be a point $(y, z) \in Z^2$ such that $Z^2 \setminus \{(y, z)\}$ is not connected. However, this is not possible because Z is connected, a contradiction. Thus, any connected component of X is a singleton. Now let us observe that X is naturally embedded in X^2, so X also has a continuous weak selection. According to [5, Theorem 4.1], this implies that the connected components of X coincide with the quasi-components of X. Hence, X is totally disconnected. □

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