CONTINUITY OF THE MODULUS OF CONTINUITY

Valentin G. Gutev

Department of Mathematics, University of Sofia, Sofia, Bulgaria

April 1, 1994

Preliminary version

Let (X, d) and (Y, d) be metric spaces, and let $C(X, Y)$ be the set of all continuous maps from X to Y, endowed with the topology of uniform convergence.

Recall that $f \in C(X, Y)$ if for every $x \in X$ and every $\varepsilon > 0$ there exists a $\delta(f, x, \varepsilon) > 0$ such that

$$x' \in X \text{ and } d(x, x') < \delta(f, x, \varepsilon) \quad \text{implies} \quad \rho(f(x), f(x')) < \varepsilon \ .$$

In [2, Theorem 1.1] it was proved that this $\delta(f, x, \varepsilon)$ can be assumed to be continuously depending on the triple (f, x, ε) provided X is locally compact. In the present note we show that the last assumption on X to be locally compact is superfluous. Namely, the following theorem holds true.

Theorem 1. *Let (X, d) and (Y, ρ) be metric spaces. Then there exists a continuous function*

$$\delta \colon C(X, Y) \times X \times (0, +\infty) \to (0, +\infty)$$

such that, for every $(f, x, \varepsilon) \in C(X, Y) \times X \times (0, +\infty)$ and every $x' \in X$,

$$d(x, x') < \delta(f, x, \varepsilon) \quad \textit{implies} \quad \rho(f(x), f(x')) < \varepsilon \ .$$

1991 *Mathematics Subject Classification.* 26A15, 54C05, 54C60, 54C65.

Key words and phrases. modulus of continuity, set-valued mapping, selection.

Remark. Theorem 1 answers in negative a question raised by D. Repovš and P. Semenov [2, Question 2.1].

Turning to the proof of Theorem 1 we need some set-valued mappings terminology and notation. Whenever E is a linear normed space, we write:

$$2^E = \{S \subset E : S \neq \emptyset\} \quad \text{and} \quad \mathcal{K}(E) = \{S \in 2^E : S \text{ is convex}\} .$$

Suppose Z is a topological space. A set-valued mapping $\varphi\colon X \to 2^E$ is *lower semi-continuous*, or l.s.c., if $\varphi^{-1}(U) = \{z \in Z : \varphi(x) \cap U \neq \emptyset\}$ is open in Z for every open $U \subset E$. A map $h\colon Z \to 2^E$ is a *selection* for $\varphi\colon Z \to 2^E$ if $h(z) \in \varphi(z)$ for every $z \in Z$. Finally, whenever (Z, p) is a metric space, $z \in Z$ and $\mu > 0$, we will also use $B_\mu(z)$ to denote the μ-neighbourhood of z, i.e. $B_\mu(z) = \{z' \in Z : p(z, z') < \mu\}$.

The proof of Theorem 1 follows that of [2, Theorem 1.1]. Let us be more specific. The function δ of [2, Theorem 1.1] was constructed as a continuous selection for an l.s.c. mapping $\Delta\colon C(X,Y) \times X \times (0, +\infty) \to \mathcal{K}(0, +\infty)$, and that was done by using one of E. Michael selection theorems [3, Theorem 3.1‴]. Our function δ will be constructed as a continuous selection for a set-valued mapping $\Delta\colon C(X,Y) \times X \times (0, +\infty) \to \mathcal{K}(0, +\infty)$ too. However, our set-valued mapping Δ fails to be l.s.c. In fact, this mapping appears in [2] too and is defined by the formula:

$$\Delta(f, x, \varepsilon) = \{r > 0 : x' \in X, d(x, x') < r \text{ implies } \rho(f(x), f(x') < \varepsilon\} . \tag{2}$$

In case $X = Y = (-1, 1)$, as shows [2, Example], this Δ is not l.s.c.

By this reason, we will apply another selection result instead of [3, Theorem 3.1‴]. This result deals with set-valued mappings with the following property: Suppose $\mathcal{S}$ is a family of subsets of a space Z; It is called that a set-valued mapping $\varphi\colon Z \to 2^E$ is *$\mathcal{S}$-fixed* [1] if $\bigcap\{\varphi(z) : z \in S\} \neq \emptyset$ for every $S \in \mathcal{S}$. Our selection result reads now as follows.

Lemma 3. *Let Z be a paracompact space, E be a linear normed space, and let $\varphi\colon Z \to \mathcal{K}(E)$ be $\mathcal{V}$-fixed for some open cover $\mathcal{V}$ of Z. Then φ admits a single-valued continuous selection.*

Proof. Define a map $e\colon \mathcal{V} \to E$ such that $e(V) \in \bigcap\{\varphi(z) : z \in V\}$ for every $V \in \mathcal{V}$. Since Z is paracompact, there exists a locally-finite partition $\{h_V : V \in \mathcal{V}\}$ of unity on Z index-subordinated to $\mathcal{V}$. Let us finally check that $f\colon Z \to E$, defined by $f(z) = \sum\{h_V(z)e(V) : V \in \mathcal{V}\}$, is the required one. Since f is clearly continuous, we have only to check that f is a selection for φ. So, let $z \in Z$ and let $\mathcal{V}_z = \{V \in \mathcal{V} : z \in V\}$. Then $V \in \mathcal{V}_z$ implies $e(V) \in \varphi(z)$, and therefore

$$f(z) = \sum\{h_V(z)e(V) : V \in \mathcal{V}\} = \sum\{h_V(z)e(V) : V \in \mathcal{V}_z\} \in \varphi(z)$$

because $\varphi(z)$ is convex. □

Having already established Lemma 3, we now complete the proof of Theorem 1 by proving the following simple lemma.

Lemma 4. *Under the assumptions of Theorem 1, the set-valued mapping* $\Delta\colon C(X,Y) \times X \times (0,+\infty) \to \mathcal{K}(0,+\infty)$ *defined by (2) is* $\mathcal{V}$*-fixed for some open cover* $\mathcal{V}$ *of* $C(X,Y) \times X \times (0,+\infty)$.

Proof. Suppose $z = (f, x, \varepsilon) \in C(X,Y) \times X \times (0,+\infty)$. It suffices to show that there exists a neighbourhood $V(z)$ of z such that

$$\bigcap\{\Delta(z') : z \in V(z)\} \neq \emptyset\ .$$

To this end, let

$$\alpha = \frac{\varepsilon}{18}\ , \qquad \gamma = \frac{\varepsilon}{2} \qquad \text{and} \qquad \lambda = \frac{2\varepsilon}{3}\ . \tag{5}$$

Since f is continuous at x, there exists an $r > 0$ such that, for every $x' \in X$,

$$d(x,x') < 2r \quad \text{implies} \quad \rho(f(x), f(x')) < \frac{\alpha}{2}\ .$$

Note this r has the following property

$$x',\ x'' \in B_{2r}(x) \quad \text{implies} \quad \rho(f(x'), f(x'')) < \alpha\ . \tag{6}$$

Let us finally check that

$$V(z) = B_\alpha(f) \times B_r(x) \times B_\gamma(\lambda) \tag{7}$$

satisfies all our requirements. Indeed, by (5),

$$|\varepsilon - \lambda| = \varepsilon - \frac{2\varepsilon}{3} = \frac{\varepsilon}{3} < \frac{\varepsilon}{2} = \gamma\ .$$

So, $z \in V(z)$. As for $\bigcap\{\Delta(z') : z \in V(z)\} \neq \emptyset$, we show that by proving r belongs to this intersection. Let $z' = (g, x', \varepsilon') \in V(z)$ and let $x'' \in B_r(x')$. By (6), $\rho(f(x'), f(x'')) < \alpha$ because $x',\ x'' \in B_{2r}(x)$. Then, by (7),

$$\begin{aligned}\rho(g(x'), g(x'')) &\leq \rho(g(x'), f(x')) + \rho(f(x'), f(x'')) + \rho(f(x''), g(x'')) \\ &< 3\alpha = \frac{\varepsilon}{6}\ .\end{aligned}$$

On the other hand,

$$\lambda - \gamma = \frac{2\varepsilon}{3} - \frac{\varepsilon}{2} = \frac{\varepsilon}{6}$$

and therefore $\rho(g(x'), g(x'')) < 3\alpha = \lambda - \gamma < \varepsilon'$ because $\varepsilon' \in B_\gamma(\lambda)$. $\square$

References

1. I. Blum and S. Swaminathan, *Continuous selections and realcompactness*, Pacific J. Math. **93** (1981), 251–260.
2. D. Repovš and P. Semenov, *An Application of the theory of selections in analysis*, University of Ljubljana, Preprint series **30** (1992), 392.
3. E. Michael, *Continuous selections I*, Ann. Math. **63** (1956), 361–382.

Bulgarian Academy of Sciences, Institute of Mathematics, 1090 Sofia, Bulgaria

E-mail address: `Gutev@bgcict.bitnet`