

APPROACHING POINTS BY CONTINUOUS SELECTIONS

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Revised for a possible acceptance in J. Math. Soc. Japan

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Abstract. Some further results about special Vietoris continuous selections and totally disconnected spaces are obtained, also several applications are demonstrated. In particular, it is demonstrated that a homogeneous separable metrizable space has a continuous selection for its Vietoris hyperspace if and only if it is discrete, or a discrete sum of copies of the Cantor set, or is the irrational numbers.

1. Introduction

Let X be a topological space, and let $\mathcal{F}(X)$ be the set of all non-empty closed subsets of X . A map $f : \mathcal{F}(X) \rightarrow X$ is a *selection* for $\mathcal{F}(X)$ if $f(S) \in S$ for every $S \in \mathcal{F}(X)$. A selection $f : \mathcal{F}(X) \rightarrow X$ is *continuous* if it is continuous with respect to the Vietoris topology τ_V on $\mathcal{F}(X)$. Let us recall that τ_V is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \subset \bigcup \mathcal{V} \text{ and } S \cap V \neq \emptyset, \text{ whenever } V \in \mathcal{V} \right\},$$

where \mathcal{V} runs over the finite families of open subsets of X . Sometimes, for reasons of convenience, we shall say that f is *Vietoris continuous* to stress the attention that f is continuous with respect to the topology τ_V . Finally, for a space X , we let $\mathcal{Sel}(X)$ to be the set of all Vietoris continuous selections for $\mathcal{F}(X)$.

In the sequel, all spaces are assumed to be at least Hausdorff. This paper was inspired by the following two results about Vietoris continuous selections and disconnectedness-like properties.

Theorem 1.1 ([8]). *If X is a first countable space, with $\mathcal{Sel}(X) \neq \emptyset$, then it is zero-dimensional if and only if for every point $x \in X$ there exists an $f_x \in \mathcal{Sel}(X)$ such that $f_x^{-1}(x) = \{S \in \mathcal{F}(X) : x \in S\}$.*

Theorem 1.2 ([8]). *If X is a space such that $\{f(X) : f \in \mathcal{Sel}(X)\}$ is dense in X , then X is totally disconnected.*

Date: September 5, 2005.

2000 Mathematics Subject Classification. 54B20, 54C65.

Key words and phrases. Hyperspace topology, Vietoris topology, continuous selection.

This research is supported in part by the National Research Foundation of South Africa under Grant number 2053735.

Here, X is *zero-dimensional* if it has a base of clopen sets (i.e., if $\text{ind}(X) = 0$), and X is *totally disconnected* if any two points of X can be separated by clopen sets.

Theorem 1.1 was naturally generalized in [10]. On the other hand, it is still an open question if “totally disconnected” in the conclusion of Theorem 1.2 can be strengthened to “zero-dimensional”, see [8, 9].

The purpose of this paper is to establish some further results about spaces X for which $\{f(X) : f \in \mathcal{S}el(X)\}$ is dense in X . In the first place, it is shown that $\{f(X) : f \in \mathcal{S}el(X)\}$ is dense in X if and only if X has a clopen π -base and $\mathcal{S}el(X) \neq \emptyset$, see Theorem 2.1. Several applications follow by this characterization. For instance, if X is metrizable and $\{f(X) : f \in \mathcal{S}el(X)\}$ is dense in X , then the set of all points at which X is zero-dimensional must be also dense in X , see Corollary 3.1. Another natural situation to apply this characterization is for homogeneous spaces. Namely, a metrizable homogeneous space X which has a continuous selection for $\mathcal{F}(X)$ must be zero-dimensional, see Corollary 3.2. Finally, we characterize all separable metrizable homogeneous spaces which have a Vietoris continuous selection (Corollary 3.3), also all locally compact topological groups with this property (Corollary 3.4).

In the second place, it is provided another characterization of spaces X with the property that $D = \{f(X) : f \in \mathcal{S}el(X)\}$ is dense in X . It is based on special members of the set D , which are called *countably-approachable* points, see Section 4. The idea of this is somehow related to Theorem 1.1, and demonstrates that if there is a “countable approach” to a point $p \in X$, then one can construct a continuous selection $f \in \mathcal{S}el(X)$, with $f(X) = p$, see Theorem 4.1. Such countably-approachable points can be useful to show, for instance, that, for a regular first countable space X , the set $\{f(X) : f \in \mathcal{S}el(X)\}$ is dense in X if and only if it coincides with X , see Corollary 4.5.

2. Selections and π -bases

A π -base for a space X is a collection \mathcal{P} of open subsets such that every non-empty open subset $U \subset X$ contains some non-empty $V \in \mathcal{P}$.

Theorem 2.1. *If X is a space, with $\mathcal{S}el(X) \neq \emptyset$, then $\{f(X) : f \in \mathcal{S}el(X)\}$ is dense in X if and only if X has a clopen π -base.*

The proof of Theorem 2.1 is based on the following observation.

Lemma 2.2. *Let X be a space, f be a continuous selection for $\mathcal{F}(X)$, $S \in \mathcal{F}(X)$, and let G be a clopen neighbourhood of $f(S)$. Then, there exists a finite cover \mathcal{W} of S consisting of non-empty clopen subsets of X such that $f(\langle \mathcal{W} \rangle) \subset G$.*

Proof. Let $\mathcal{H} \subset \mathcal{F}(X)$ be a chain in $f^{-1}(G)$ which contains S , and which is maximal with respect to the usual set-theoretical inclusion. Since $f^{-1}(G)$ is a τ_V -clopen set, by a result of [5] (see, also, [3, 7]), $H = \max \mathcal{H}$ is a clopen subset of X , and clearly $H \in f^{-1}(G)$. Then, in particular, there exists a finite family \mathcal{U} of non-empty open subsets of X such that $H \in \langle \mathcal{U} \rangle \subset f^{-1}(G)$. Since $H = \max \mathcal{H}$, it now follows that $H = \bigcup \mathcal{U}$. In what follows, let us recall that a cover \mathcal{W} of H is called *irreducible* if $H \neq \bigcup \{W \in \mathcal{W} : W \neq W_0\}$ for every $W_0 \in \mathcal{W}$. Take an irreducible subcover $\mathcal{W} \subset \mathcal{U}$ of H (see, for instance, [6, Theorem 5.3.1]), and let us show that \mathcal{W} is as required. To this end, we first show that $\langle \mathcal{W} \rangle$ is a τ_V -clopen set. Take an $F \in \overline{\langle \mathcal{W} \rangle}$, and let V be a neighbourhood of $f(F \cup H)$. Note that $F \cup H \in \overline{\langle \mathcal{W} \rangle}$ because $H \in \langle \mathcal{W} \rangle \subset \overline{\langle \mathcal{W} \rangle}$. Since f is continuous, this now implies the existence of $T \in f^{-1}(V) \cap \langle \mathcal{W} \rangle$. Then, $f(T) \in V \cap G$, and therefore $f(F \cup H) \in \overline{G} = G$ because V was an arbitrary neighbourhood of $f(F \cup H)$. Thus, $F \cup H \in \mathcal{H}$ because \mathcal{H} is a maximal chain in $f^{-1}(G)$, hence $F \subset H$. That is, $\overline{\langle \mathcal{W} \rangle} = \langle \mathcal{W} \rangle$.

Clearly $S \subset H = \bigcup \mathcal{W}$, so it only remains to show that \mathcal{W} consists of clopen sets. Suppose if possible that there exists $W_0 \in \mathcal{W}$ which is not closed. Then $F = \bigcup \mathcal{W} \setminus W_0$ is a non-empty closed subset of X because $\bigcup \mathcal{W} = H$ is clopen. Take a basic τ_V -neighbourhood $\langle \mathcal{V} \rangle$ of F . Then, $\langle \mathcal{V} \rangle \cap \langle \mathcal{W} \rangle \neq \emptyset$. Indeed, $(\bigcup \mathcal{V}) \cap W_0 \neq \emptyset$ because W_0 is not closed, while $F \cap W \neq \emptyset$ for every $W \in \mathcal{W} \setminus W_0$, because \mathcal{W} is an irreducible cover of H . Therefore, $F \cup \{t\} \in \langle \mathcal{V} \rangle \cap \langle \mathcal{W} \rangle$ for some (every) point $t \in (\bigcup \mathcal{V}) \cap W_0$. As a result, we get that $F \in \overline{\langle \mathcal{W} \rangle} = \langle \mathcal{W} \rangle$. However, this is impossible because $F \cap W_0 = \emptyset$. \square

Proof of Theorem 2.1. Suppose that $g \in \text{Sel}(X)$, and that \mathcal{P} is a clopen π -base for X . Next, take a non-empty open subset $U \subset X$. Then, there exists a non-empty clopen set $V \in \mathcal{P}$ such that $V \subset U$. Now, we can repeat some of the arguments in the proof of [8, Lemma 2.1]. Namely, we set

$$\mathcal{V}_0 = \{S \in \mathcal{F}(X) : S \cap V = \emptyset\},$$

and

$$\mathcal{V}_1 = \{S \in \mathcal{F}(X) : S \cap V \neq \emptyset\}.$$

Thus, we get a τ_V -clopen partition $\{\mathcal{V}_0, \mathcal{V}_1\}$ of $\mathcal{F}(X)$. Then, we can define a selection f for $\mathcal{F}(X)$ by letting $f(S) = g(S)$ if $S \in \mathcal{V}_0$, and $f(S) = g(S \cap V)$ otherwise. Clearly, f is τ_V -continuous and $f(X) = g(X \cap V) \in V \subset U$. So, $\{f(X) : f \in \text{Sel}(X)\}$ is dense in X .

To prove the converse, suppose that $\{f(X) : f \in \text{Sel}(X)\}$ is dense in X , and take an open subset $U \subset X$ such that $U \neq \emptyset \neq X \setminus U$. Then, by hypothesis, there exists a selection $f \in \text{Sel}(X)$, with $f(X) \in U$. Since f is continuous, there exists a finite cover \mathcal{V} of X consisting of non-empty open subsets such that $f(\langle \mathcal{V} \rangle) \subset U$.

Let $S = X \setminus U$, and let $F \subset U$ be a non-empty finite set such that $S \cup F \in \langle \mathcal{V} \rangle$. Observe that $f(S \cup F) \in F$. On the other hand, by Theorem 1.2, X is totally disconnected and $f(S) \notin F$. Hence, there exists a clopen set $G \subset X$ such that $f(S) \in G$ and $G \cap F = \emptyset$. Then, by Lemma 2.2, there exists a finite family \mathcal{W} of non-empty clopen subsets of X such that $S \subset \bigcup \mathcal{W}$ and $f(\langle \mathcal{W} \rangle) \subset G$. Now we have either $S \notin \langle \mathcal{W} \rangle$ or $S \in \langle \mathcal{W} \rangle$. If $S \notin \langle \mathcal{W} \rangle$, then there should exist $W_0 \in \mathcal{W}$ such that $W_0 \cap S = \emptyset$ because $S \subset \bigcup \mathcal{W}$. Then, W_0 is a non-empty clopen subset, with $W_0 \subset X \setminus S = U$. Suppose finally that $W \cap S \neq \emptyset$ for every $W \in \mathcal{W}$. In this case, $F \setminus \bigcup \mathcal{W} \neq \emptyset$. Indeed, if $F \subset \bigcup \mathcal{W}$, then this will imply that $S \cup F \in \langle \mathcal{W} \rangle$, so $f(S \cup F) \in G$. However, by construction, $f(S \cup F) \in F$ and $F \cap G = \emptyset$. The contradiction so obtained demonstrates that, in this case, $T = X \setminus \bigcup \mathcal{W}$ is a non-empty clopen subset, with $T \subset X \setminus S = U$. This completes the proof. \square

It should be mentioned that the proof of Theorem 2.1 relies on Theorem 1.2, hence on the total-disconnectedness of X . Nevertheless, it seems justifiable to mention the following immediate consequence of Theorems 2.1 and 1.2.

Corollary 2.3. *Let X be a space with a clopen π -base and $\text{Sel}(X) \neq \emptyset$. Then, X is totally-disconnected.*

3. Many selections and metrizable spaces

Let us recall that a regular space X is *Moore* if there is a sequence $\{\mathcal{W}_k : k < \omega\}$ of open covers of X such that $\{\text{St}(x, \mathcal{W}_k) : k < \omega\}$ is a local base at x for every $x \in X$. Here, $\text{St}(x, \mathcal{W}_k) = \bigcup \{W \in \mathcal{W}_k : x \in W\}$.

Also, for a space X and $x \in X$, we will write that $\text{ind}_x(X) = 0$ if X has a clopen base at x , i.e. if it is *zero-dimensional* at x .

Corollary 3.1. *If X is a Moore space, with $\text{Sel}(X) \neq \emptyset$, then the set $\{f(X) : f \in \text{Sel}(X)\}$ is dense in X if and only if the set $\{x \in X : \text{ind}_x(X) = 0\}$ is dense in X .*

Proof. According to Theorem 2.1, we have to show that $\{x \in X : \text{ind}_x(X) = 0\}$ is dense in X if and only if X has a clopen π -base. So, suppose that \mathcal{P} is a clopen π -base for X , and let $\{\mathcal{W}_k : k < \omega\}$ be as in the definition of a Moore space. Then, for every $k < \omega$ there exists a pairwise disjoint family $\mathcal{P}_k \subset \mathcal{P}$ such that \mathcal{P}_k refines \mathcal{W}_k , and $G_k = \bigcup \mathcal{P}_k$ is dense in X . By a result of [11], X is a Baire space because it is regular and has a continuous selection. Then, $G = \bigcap \{G_k : k < \omega\}$ is a dense G_δ -subset of X , and clearly $\text{ind}_x(X) = 0$ for every $x \in G$. Thus $\{x \in X : \text{ind}_x(X) = 0\}$ is dense in X because it contains G . To show the converse, for every $x \in X$, with $\text{ind}_x(X) = 0$, take a clopen base \mathcal{P}_x at x . Then, $\mathcal{P} = \bigcup \{\mathcal{P}_x : \text{ind}_x(X) = 0\}$ is a clopen π -base for X because $\{x \in X : \text{ind}_x(X) = 0\}$ is dense in X . \square

Corollary 3.2. *If X is a homogeneous Moore space, with $\text{Sel}(X) \neq \emptyset$, then it is zero-dimensional.*

Proof. Since X is homogeneous, $X = \{f(X) : f \in \text{Sel}(X)\}$. Hence, by Corollary 3.1, $\{x \in X : \text{ind}_x(X) = 0\}$ is dense in X , and, in particular, non-empty. So, X has at least one point at which it is zero-dimensional, hence it is zero-dimensional because it is homogeneous. \square

Corollary 3.3. *Let X be a homogeneous separable metrizable space such that $\text{Sel}(X) \neq \emptyset$. Then, one of the following holds:*

- (a) X is a discrete space,
- (b) X is a discrete sum of copies of the Cantor set,
- (c) X is the irrational line.

Proof. If X contains an isolated point, then all points of X must be isolated because it is homogeneous. Hence, in this case, X is discrete. Suppose that X contains a non-isolated point, then X must be dense in itself because it is homogeneous. We distinguish the following two cases. If X contains a clopen compact subset, then it has a discrete cover \mathcal{C} of clopen compact sets. Note that X is separable, hence its covering dimension is zero because, by Corollary 3.2, it is zero-dimensional. Further, let us observe that each $C \in \mathcal{C}$ is a zero-dimensional compact metric space, which is dense in itself because X is dense in itself. Hence, by a result of [4], C is homeomorphic to the Cantor set. Thus, (b) holds in this case. Finally, let us suppose that X doesn't contain any non-empty compact open subset. According to a result of [12], X must be completely metrizable because it has a Vietoris continuous selection, while, by Corollary 3.2, it is zero-dimensional. So, by a result of [1], X is homeomorphic to the irrational numbers. \square

A space X is *orderable* (or, *linearly orderable*) if there exists a linear order \leq on X such that the sets $\{y \in Y : x < y\}$ and $\{y \in X : y < x\}$ constitute a subbase for the topology of X . A topological group G is called *topologically orderable* if it is an orderable topological space (no relation between the group operations and the order is assumed).

Corollary 3.4. *A locally compact topological group G is totally disconnected and topologically orderable if and only if $\text{Sel}(G) \neq \emptyset$.*

Proof. Suppose that G is totally disconnected and orderable. Then, by [15, Theorem 5.5] (see, also, [14, Theorem 9]), G is either discrete or it contains a clopen subgroup homeomorphic to the Cantor set. In both cases, G is represented as a discrete sum of spaces G_α , with $\text{Sel}(G_\alpha) \neq \emptyset$. Hence, G itself has a continuous selection $f : \mathcal{F}(G) \rightarrow G$.

Suppose now that $\text{Sel}(G) \neq \emptyset$. Then, $G = \{f(G) : f \in \text{Sel}(G)\}$ because G is homogeneous, so, by Theorem 1.2, G is totally disconnected. Then, G has a basis

of neighbourhoods at the identity consisting of compact open subgroups, see [13]. Hence, G contains a compact open subgroup H . Then, either H is finite or it is infinite. Also, let us observe that $\text{Sel}(H) \neq \emptyset$ because $H \in \mathcal{F}(G)$. In case H is infinite, by [10, Corollary 5.6] (see, also, [2, Corollary 1.27]), H is homeomorphic to the Cantor set. Thus, H is either finite or homeomorphic to the Cantor set. Hence, by [15, Theorem 5.5], G is topologically orderable. \square

4. Countably-approachable points

In the present section we provide a possible variant of Corollary 3.1 for arbitrary spaces. To this end, let us say that a point $p \in X$ is *0-approachable* if p is an isolated point of X , and that p is *ω -approachable* if there exists an open subset $U \subset X \setminus \{p\}$ such that $\bar{U} = U \cup \{p\}$, and p has a countable clopen base in \bar{U} . Let us observe that if p is *ω -approachable*, and if $\{W_n : n < \omega\}$ is a strictly decreasing clopen base at p in \bar{U} , then $S_n = W_n \setminus W_{n+1}$, $n < \omega$, is a disjoint family of non-empty clopen subset of X such that $p \notin S_n$ for every $n < \omega$, and $\{S_n : n < \omega\}$ is τ_V -convergent to p . One can easily see that the converse is also true, so, in the sequel, we will mainly rely on this characterization of ω -approachable points.

In what follows, we say that $p \in X$ is *countably-approachable* if p is either 0-approachable or it is ω -approachable.

Theorem 4.1. *For a space X , with $\text{Sel}(X) \neq \emptyset$, the following are equivalent:*

- (a) *The set $\{f(X) : f \in \text{Sel}(X)\}$ is dense in X .*
- (b) *The set of all countably-approachable points of X is dense in X .*

The proof of Theorem 4.1 consists of the following separate observations.

Lemma 4.2. *Let X be a space, with $\text{Sel}(X) \neq \emptyset$, and let $p \in X$ be a countably-approachable point of X . Then, there exists an $f \in \text{Sel}(X)$, with $f(X) = p$.*

Proof. If p is an isolated point of X , then this follows by [8, Lemma 2.1]. So, suppose that p is an ω -approachable point of X , and let $\{S_n : n < \omega\}$ be a disjoint family of non-empty clopen subsets of X such that $p \notin S_n$ for every $n < \omega$, and $\{S_n : n < \omega\}$ is τ_V -convergent to p . Following [8, Lemma 2.3], we let

$$\mathcal{V}_0 = \{F \in \mathcal{F}(X) : F \cap S_0 = \emptyset\},$$

and

$$\mathcal{V}_1 = \{F \in \mathcal{F}(X) : F \cap S_0 \neq \emptyset\}.$$

Thus, we get a τ_V -clopen partition of $\mathcal{F}(X)$ because S_0 is a clopen set. Then, consider the sets

$$\mathcal{V}_1^0 = \{F \in \mathcal{V}_1 : S_n \cap F = \emptyset \text{ for some } n < \omega\},$$

and

$$\mathcal{V}_1^1 = \{F \in \mathcal{V}_1 : S_n \cap F \neq \emptyset \text{ for every } n < \omega\}.$$

For later use, let us observe that $F \in \mathcal{V}_1^1$ implies $p \in F$.

Now, for every $F \in \mathcal{V}_1^0$, let $n(F) = \min\{n < \omega : S_{n+1} \cap F = \emptyset\}$. Next, take a selection $g \in \text{Sel}(X)$, and then define another selection $f : \mathcal{F}(X) \rightarrow X$ by letting $f \upharpoonright \mathcal{V}_0 = g \upharpoonright \mathcal{V}_0$, while $f(F) = g(S_{n(F)} \cap F)$ if $F \in \mathcal{V}_1^0$, and $f(F) = p$ otherwise. Since $X \in \mathcal{V}_1^1$, we have that $f(X) = p$, hence it only remains to show that f is continuous. Since \mathcal{V}_0 is τ_V -clopen and g is continuous, it now suffices to show that $f \upharpoonright \mathcal{V}_1$ is continuous, so take an $F \in \mathcal{V}_1$. We distinguish the following two cases. If $F \in \mathcal{V}_1^0$, then $F \cap S_k \neq \emptyset$ for every $k \leq n(F)$, and $F \cap S_{n(F)+1} = \emptyset$. On the other hand, $f(F) = g(S_{n(F)} \cap F) \in S_{n(F)}$, while $S_{n(F)}$ is clopen. Then, consider the τ_V -clopen set $\langle \mathcal{U} \rangle$, where

$$\mathcal{U} = \{S_k : k \leq n(F)\} \cup \{X \setminus S_{n(F)+1}\}.$$

Note that $F \in \langle \mathcal{U} \rangle \subset \mathcal{V}_1^0$, while the map $\varphi : \langle \mathcal{U} \rangle \rightarrow \mathcal{F}(S_{n(F)})$, defined by $\varphi(T) = T \cap S_{n(F)}$, is τ_V -continuous. Also, $T \in \langle \mathcal{U} \rangle$ implies $n(T) = n(F)$, hence $f \upharpoonright \langle \mathcal{U} \rangle = g \circ \varphi$. Thus, f is continuous at F because so are g and φ . Finally, let us consider the case when $F \in \mathcal{V}_1^1$. By definition, $f(F) = p$, while $F \cap S_k \neq \emptyset$ for every $k < \omega$. Take a neighbourhood V of p in X . Then, there exists an $m < \omega$, with $S_n \subset V$ for every $n \geq m$. In this case, let

$$\mathcal{U} = \{S_k : k \leq m\} \cup \{X\}.$$

Thus, we get a τ_V -neighbourhood of F such that $f(\langle \mathcal{U} \rangle) \subset V$. Indeed, take a $T \in \langle \mathcal{U} \rangle$, and then observe that $T \in \mathcal{V}_1$ because $T \cap S_0 \neq \emptyset$. If $T \cap S_k \neq \emptyset$ for every $k < \omega$, then $f(T) = p \in V$. If $T \cap S_k = \emptyset$ for some $k < \omega$, then $n(T) \geq m$, so $f(T) = g(S_{n(T)} \cap T) \in S_{n(T)} \subset V$. \square

Proposition 4.3. *Let $p \in X$ be a non-isolated point of X , and let $f \in \text{Sel}(X)$ be such that $f(X) = p$. Then, for every closed subset $F \subset X$, with $p \notin F$, there exists a closed subset $T \subset X$ such that $F \subset T$, $p \notin T$, and $f(T \cup \{x\}) = x$ for some $x \in X \setminus T$.*

Proof. Let $F \subset X$ be as in this statement. Then, $U = X \setminus F$ is a neighbourhood of p , so there exists a finite open cover \mathcal{W} of X such that $X \in \langle \mathcal{W} \rangle$ and $f(\langle \mathcal{W} \rangle) \subset U$. Since p is a non-isolated point of U , there now exists a finite set $S \subset U \setminus \{p\}$ such that $F \cup S \in \langle \mathcal{W} \rangle$. Then, we can take $T = (F \cup S) \setminus \{f(F \cup S)\}$, which works because $f(F \cup S) \in S$. \square

We finalize the proof of Theorem 4.4 with the following lemma.

Lemma 4.4. *Let X be a space such that $\{f(X) : f \in \text{Sel}(X)\}$ is dense in X . Then, every non-empty open subset of X contains a countably-approachable point.*

Proof. Let $U \subset X$ be a non-empty clopen set. If U contains some isolated point, then clearly U contains a countably-approachable point as well. So, suppose that U has no isolated points, and then set $F_0 = X \setminus U$. Also, take an $f \in \text{Sel}(X)$ such that $p = f(X) \in U$. According to Proposition 4.3, there now exists a closed subset $T_0 \subset X$ such that $F_0 \subset T_0$, $p \notin T_0$, and $f(T_0 \cup \{x\}) = x$ for some $x \notin T_0$. Since f is continuous, we may find an open set $U_0 \subset X \setminus T_0 \subset U$ such that $f(T_0 \cup \{x\}) = x$ for every $x \in U_0$. Then, $|U_0| > 1$ because U does not have any isolated point, so, by Theorem 2.1, $U_0 \setminus \{p\}$ contains a non-empty clopen subset S_0 . Thus, in fact, we have constructed a closed set $T_0 \subset X$, with $p \notin T_0$, and a non-empty clopen subset $S_0 \subset X$ such that $F_0 \subset T_0 \subset X \setminus S_0$, $p \notin S_0$, and $f(T_0 \cup \{x\}) = x$ for every $x \in S_0$. Now, we can set $F_1 = T_0 \cup S_0$, and we can repeat the same arguments. Hence, by induction, we get an increasing sequence $\{T_n : n < \omega\}$ of closed subsets, and a disjoint family $\{S_n : n < \omega\}$ of non-empty clopen subsets such that, for every $n < \omega$,

- (a) $T_n \cup S_n \subset T_{n+1} \subset X \setminus (S_{n+1} \cup \{p\})$,
- (b) $f(T_n \cup \{x\}) = x$, for every $x \in S_n$.

Set $T = \bigcup\{T_n : n < \omega\}$, $S = \bigcup\{S_n : n < \omega\}$, and $q = f(\overline{T})$. We are going to show that q is an ω -approachable point. Towards this end, let us observe that if $y_n \in S_n$ for every $n < \omega$, then $q = \lim_{n \rightarrow \infty} y_n$. Indeed, in this case, by (a), we have that $T_n \subset T_n \cup \{y_n\} \subset T_{n+1}$, so, by (b),

$$(4.1) \quad q = \lim_{n \rightarrow \infty} f(T_{n+1}) = \lim_{n \rightarrow \infty} f(T_n \cup \{y_n\}) = \lim_{n \rightarrow \infty} y_n.$$

In particular, this implies that $q \notin S$ because $\{S_n : n < \omega\}$ is a disjoint open family. Also, $\{S_n : n < \omega\}$ is τ_V -convergent to q . Indeed, if this fails, then there should be some neighbourhood W of q in X so that $S_n \setminus W \neq \emptyset$ for infinitely many $n < \omega$. Hence, we can find a strictly increasing sequence $\{n_k : k < \omega\} \subset \omega$, and a sequence of points $\{y_k : k < \omega\}$ so that $y_k \in S_{n_k} \setminus W$ for every $k < \omega$. According to (4.1), this will imply that $q = \lim_{k \rightarrow \infty} y_k \in X \setminus W$, which is clearly impossible. Thus, $q \in \overline{U} = U$ is an ω -approachable point of X . Since, by Theorem 2.1, X has a clopen π -base, this completes the proof. \square

The following is now an immediate consequence of Lemma 4.2.

Corollary 4.5. *If X is a regular first countable space and $\{f(X) : f \in \text{Sel}(X)\}$ is dense in X , then $X = \{f(X) : f \in \text{Sel}(X)\}$.*

Proof. Take a non-isolated point $p \in X$, and let $\{W_n : n < \omega\}$ be an open local base at p such that $\overline{W_{n+1}} \subset W_n$ and $C_n = W_n \setminus \overline{W_{n+1}} \neq \emptyset$ for every $n < \omega$. According to Theorem 2.1, every C_n contains a non-empty clopen subset $S_n \subset C_n$. Then, the sequence $\{S_n : n < \omega\}$ is τ_V -convergent to p , so p is an ω -approachable point. \square

ACKNOWLEDGEMENT. The author would like to express his best gratitude to the referee for several valuable remarks.

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