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## FELL-CONTINUOUS SELECTIONS AND TOPOLOGICALLY WELL-ORDERABLE SPACES II

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ABSTRACT. The present paper improves a result of [3] by showing that a space  $X$  is topologically well-orderable if and only if there exists a selection for  $\mathcal{F}_2(X)$  which is continuous with respect to the Fell topology on  $\mathcal{F}_2(X)$ . In particular, this implies that  $\mathcal{F}(X)$  has a Fell-continuous selection if and only if  $\mathcal{F}_2(X)$  has a Fell-continuous selection.

### 1. INTRODUCTION

Let  $X$  be a topological space, and let  $\mathcal{F}(X)$  be the family of all non-empty closed subsets of  $X$ . Also, let  $\tau$  be a topology on  $\mathcal{F}(X)$  and  $\mathcal{D} \subset \mathcal{F}(X)$ . A map  $f : \mathcal{D} \rightarrow X$  is a *selection* for  $\mathcal{D}$  if  $f(S) \in S$  for every  $S \in \mathcal{D}$ . A map  $f : \mathcal{D} \rightarrow X$  is a  $\tau$ -*continuous* selection for  $\mathcal{D}$  if it is a selection for  $\mathcal{D}$  which is continuous with respect to the relative topology  $\tau$  on  $\mathcal{D}$  as a subspace of  $\mathcal{F}(X)$ .

Two topologies on  $\mathcal{F}(X)$  will play the most important role in this paper. The first one is the *Vietoris topology*  $\tau_V$  which is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \cap V \neq \emptyset, V \in \mathcal{V}, \text{ and } S \subset \bigcup \mathcal{V} \right\},$$

where  $\mathcal{V}$  runs over the finite families of open subsets of  $X$ . The other one is the *Fell topology*  $\tau_F$  which is defined by all basic Vietoris neighbourhood  $\langle \mathcal{V} \rangle$  with the property that  $X \setminus \bigcup \mathcal{V}$  is compact.

Finally, let us recall that a space  $X$  is *topologically well-orderable* (see Engelking, Heath and Michael [2]) if there exists a linear order “ $\prec$ ” on  $X$  such that  $X$  is a linear ordered topological space with respect to  $\prec$ , and every non-empty closed subset of  $X$  has a  $\prec$ -minimal element.

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Recently, the topologically well-orderable spaces were characterized in [3, Theorem 1.3] by means of Fell-continuous selections for their hyperspaces of non-empty closed subsets.

**Theorem 1.1** ([3]). *A Hausdorff space  $X$  is topologically well-orderable if and only if  $\mathcal{F}(X)$  has a  $\tau_F$ -continuous selection.*

In the present paper, we improve Theorem 1.1 by showing that one may use  $\tau_F$ -continuous selections only for the subset  $\mathcal{F}_2(X) = \{S \in \mathcal{F}(X) : |S| \leq 2\}$  of  $\mathcal{F}(X)$ . Namely, the following theorem will be proven.

**Theorem 1.2.** *A Hausdorff space  $X$  is topologically well-orderable if and only if  $\mathcal{F}_2(X)$  has a  $\tau_F$ -continuous selection.*

About related results for Vietoris-continuous selections, the interested reader is referred to van Mill and Wattel [6].

Theorem 1.2 is interesting also from another point of view. According to Theorem 1.1, it implies the following result which may have an independent interest.

**Corollary 1.3.** *If  $X$  is a Hausdorff space, then  $\mathcal{F}(X)$  has a  $\tau_F$ -continuous selection if and only if  $\mathcal{F}_2(X)$  has a  $\tau_F$ -continuous selection.*

A word should be said also about the proof of Theorem 1.2. In general, it is based on the proof of Theorem 1.1 stated in [3], and is separated in a few different steps which are natural generalizations of the corresponding ones given in [3]. In fact, the paper demonstrates that all statements of [3] remain true if  $\mathcal{F}(X)$  is replaced by  $\mathcal{F}_2(X)$ . Related to this, the interested reader may consult an alternative proof of Theorem 1.2 given in [1] and based again on the scheme in [3].

## 2. A REDUCTION TO LOCALLY COMPACT SPACES

In the sequel, all spaces are assumed to be at least Hausdorff.

In this section, we prove the following generalization of [3, Theorem 2.1].

**Theorem 2.1.** *Let  $X$  be a space such that  $\mathcal{F}_2(X)$  has a  $\tau_F$ -continuous selection. Then  $X$  is locally compact.*

**Proof.** We follow the proof of [3, Theorem 2.1]. Namely, let  $f$  be a  $\tau_F$ -continuous selection for  $\mathcal{F}_2(X)$  and suppose, if possible, that  $X$  is not locally compact. Hence, there exists a point  $p \in X$  such that  $\overline{V}$  is not compact for every neighbourhood  $V$  of  $p$  in  $X$ . Claim that there exists a point  $q \in X$  such that

$$(1) \quad q \neq p \text{ and } f(\{p, q\}) = p.$$

To this purpose, note that there exists  $F \in \mathcal{F}(X)$  such that  $F$  is not compact and  $p \notin F$ . Then,  $f^{-1}(X \setminus F)$  is a  $\tau_F$ -neighbourhood of  $\{p\}$  in  $\mathcal{F}_2(X)$ , so

there exists a finite family  $\mathcal{W}$  of open subsets of  $X$  such that  $X \setminus \bigcup \mathcal{W}$  is compact and

$$\{p\} \in \langle \mathcal{W} \rangle \cap \mathcal{F}_2(X) \subset f^{-1}(X \setminus F).$$

Then,  $F \cap W \neq \emptyset$  for some  $W \in \mathcal{W}$  because  $F$  is not compact. Therefore, there exists a point  $q \in F \cap (\bigcup \mathcal{W})$ . This  $q$  is as required.

Let  $q$  be as in (1). Since  $X$  is Hausdorff,  $f(\{q\}) \neq f(\{p, q\})$ , and  $f$  is  $\tau_F$ -continuous, there now exist two finite families  $\mathcal{U}$  and  $\mathcal{V}$  of open subsets of  $X$  such that  $X \setminus \bigcup \mathcal{U}$  is compact,  $\{q\} \in \langle \mathcal{U} \rangle$ ,  $\{p, q\} \in \langle \mathcal{V} \rangle$ , and  $\langle \mathcal{U} \rangle \cap \langle \mathcal{V} \rangle = \emptyset$ . Then,

$$(2) \quad p \in V_p = \bigcap \{V \in \mathcal{V} : p \in V\} \subset X \setminus \bigcup \mathcal{U}.$$

Indeed, suppose there is a point  $\ell \in V_p \cap (\bigcup \mathcal{U})$ . Then,  $\{\ell, q\} \in \langle \mathcal{U} \rangle$  because  $\{q\} \in \langle \mathcal{U} \rangle$ . However, we also get that  $\{\ell, q\} \in \langle \mathcal{V} \rangle$  because  $q \notin V$  for some  $V \in \mathcal{V}$  implies  $p \in V$ , hence  $\ell \in V_p \subset V$ . Thus, we finally get that  $\{\ell, q\} \in \langle \mathcal{U} \rangle \cap \langle \mathcal{V} \rangle$  which is impossible. So, (2) holds as well.

To finish the proof, it remains to observe that this contradicts the choice of  $p$ . Namely  $V_p$  becomes a neighbourhood of  $p$  which, by (2), has a compact closure because  $X \setminus \bigcup \mathcal{U}$  is compact.  $\square$

### 3. A REDUCTION TO COMPACT SPACES

For a locally compact space  $X$  we will use  $\alpha X$  to denote the one point compactification of  $X$ . For a non-compact locally compact  $X$  let us agree to denote by  $\alpha$  the point of the singleton  $\alpha X \setminus X$ .

In what follows, to every family  $\mathcal{D} \subset \mathcal{F}(X)$  we associate a family  $\alpha(\mathcal{D}) \subset \mathcal{F}(\alpha X)$  defined by

$$\alpha(\mathcal{D}) = \{S \in \mathcal{F}(\alpha X) : S \cap X \in \mathcal{D} \cup \{\emptyset\}\}.$$

The following extension theorem was actually proven in [3, Theorem 3.1].

**Theorem 3.1.** *Let  $X$  be a locally compact non-compact space  $X$ , and  $\mathcal{D} \subset \mathcal{F}(X)$ . Then,  $\mathcal{D}$  has a  $\tau_F$ -continuous selection if and only if  $\alpha(\mathcal{D})$  has a  $\tau_V$ -continuous selection  $g$  such that  $g^{-1}(\alpha) = \{\{\alpha\}\}$ .*

**Proof.** Just the same proof as in [3, Theorem 3.1] works. Namely, if  $f$  is a  $\tau_F$ -continuous selection for  $\mathcal{D}$ , we may define a selection  $g$  for  $\alpha(\mathcal{D})$  by  $g(S) = f(S \cap X)$  if  $S \cap X \neq \emptyset$  and  $g(S) = \alpha$  otherwise, where  $S \in \alpha(\mathcal{D})$ . Clearly  $g^{-1}(\alpha) = \{\{\alpha\}\}$  and, as shown in [3, Theorem 3.1],  $g$  is  $\tau_V$ -continuous. If now  $g$  is a  $\tau_V$ -continuous selection for  $\alpha(\mathcal{D})$ , with  $g^{-1}(\alpha) = \{\{\alpha\}\}$ , then  $g(S \cup \{\alpha\}) \in S$  for every  $S \in \mathcal{D}$ , so we may define a selection  $f$  for  $\mathcal{D}$  by  $f(S) = g(S \cup \{\alpha\})$ ,  $S \in \mathcal{D}$ . The verification that  $f$  is  $\tau_F$ -continuous was done in [3, Theorem 3.1].  $\square$

## 4. SPECIAL SELECTIONS AND CONNECTED SETS

In what follows, to every selection  $f : \mathcal{F}_2(X) \rightarrow X$  we associate an order-like relation “ $\prec_f$ ” on  $X$  (see Michael [5]) defined for  $x \neq y$  by

$$x_1 \prec_f x_2 \text{ iff } f(\{x_1, x_2\}) = x_1.$$

Further, we will need also the following  $\prec_f$ -intervals:

$$(x, +\infty)_{\prec_f} = \{z \in X : x \prec_f z\}$$

and

$$[x, +\infty)_{\prec_f} = \{z \in X : x \preceq_f z\}.$$

Now, we provide the generalization of [3, Theorem 4.1] for the case of  $\mathcal{F}_2(X)$ .

**Theorem 4.1.** *Let  $X$  be a space,  $a \in X$ , and let  $A \in \mathcal{F}(X)$  be a connected set such that  $|A| > 1$  and  $a \in A \cap \overline{X \setminus A}$ . Also, let  $f : \mathcal{F}_2(X) \rightarrow X$  be a  $\tau_V$ -continuous selection for  $\mathcal{F}_2(X)$ . Then,  $f^{-1}(a) \neq \{\{a\}\}$ .*

**Proof.** Suppose, on the contrary, that  $f^{-1}(a) = \{\{a\}\}$ . By hypothesis, there exists a point  $b \in A$ , with  $b \neq a$ . Since  $f$  is  $\tau_V$ -continuous,  $f(\{a, b\}) = b$  and  $a \in \overline{X \setminus A}$ , we can find a point  $c \in X \setminus A$  such that  $f(\{b, c\}) = b$ . Then,  $B = A \cap (c, +\infty)_{\prec_f}$  is a clopen subset of  $A$  because  $B = A \cap [c, +\infty)_{\prec_f}$ , see [5]. However, this is impossible because  $b \in A \setminus B$ , while  $a \in B$ .  $\square$

## 5. A FURTHER RESULT ABOUT SPECIAL SELECTIONS

Following [3], we shall say that a point  $a \in X$  is a *partition* of  $X$  if there are open subset  $L, R \subset X \setminus \{a\}$  such that  $\overline{L} \cap \overline{R} = \{a\}$  and  $L \cap R = \emptyset$ .

We finalize the preparation for the proof of Theorem 1.2 with the following result about special Vietoris continuous selections and partitions which generalizes [3, Theorem 5.1].

**Theorem 5.1.** *Let  $X$  be a compact space,  $f$  a  $\tau_V$ -continuous selection for  $\mathcal{F}_2(X)$ , and let  $a \in X$  be a partition of  $X$  such that  $f^{-1}(a) = \{\{a\}\}$ . Then,  $X$  is first countable at  $a$ .*

**Proof.** By definition, there are open sets  $L, R \subset X \setminus \{a\}$  such that  $\overline{L} \cap \overline{R} = \{a\}$  and  $L \cap R = \emptyset$ . Hence, both  $L$  and  $R$  are non-empty. Take a point  $\ell_0 \in L$ . Then, by hypothesis,  $f(\{\ell_0, a\}) = \ell_0$ . Since  $f$  is  $\tau_V$ -continuous, this implies the existence of a neighbourhood  $L_0 \subset L$  of  $\ell_0$  and a neighbourhood  $V_0$  of  $a$  such that

$$L_0 \cap V_0 = \emptyset \text{ and } f(\langle \{L_0, V_0\} \rangle \cap \mathcal{F}_2(X)) \subset L_0.$$

Since  $a \in \overline{R}$ , there exists a point  $r_0 \in V_0 \cap R$ . Observe that  $f(\{a, r_0\}) = r_0 \in V_0$ . Hence, just like before, we may find a neighbourhood  $R_0 \subset R \cap V_0$  of  $r_0$  and a neighbourhood  $W_0 \subset V_0$  of  $a$  such that

$$R_0 \cap W_0 = \emptyset \text{ and } f(\langle \{R_0, W_0\} \rangle \cap \mathcal{F}_2(X)) \subset R_0.$$

Thus, by induction, we may construct a sequence  $\{\ell_n : n < \omega\}$  of points of  $L$ , a sequence  $\{r_n : n < \omega\}$  of points of  $R$ , and open sets  $L_n, V_n, R_n, W_n \subset X$  such that

$$(3) \quad \begin{aligned} &\ell_n \in L_n, \\ &a \in V_n, \\ &L_n \cap V_n = \emptyset \text{ and} \\ &f(\langle \{L_n, V_n\} \rangle \cap \mathcal{F}_2(X)) \subset L_n, \end{aligned}$$

$$(4) \quad \begin{aligned} &r_n \in R_n, \\ &a \in W_n, \\ &R_n \cap W_n = \emptyset \text{ and} \\ &f(\langle \{R_n, W_n\} \rangle \cap \mathcal{F}_2(X)) \subset R_n, \end{aligned}$$

and

$$(5) \quad \begin{aligned} &V_{n+1} \subset W_n \subset V_n, \\ &L_{n+1} \subset L \cap W_n \text{ and} \\ &R_n \subset R \cap V_n. \end{aligned}$$

Since  $X$  is compact,  $\{\ell_n : n < \omega\}$  has a cluster point  $\ell$ , and  $\{r_n : n < \omega\}$  has a cluster point  $r$ . We claim that  $\ell = r$ . Indeed, suppose for instance that  $\ell \prec_f r$  (the case  $r \prec_f \ell$  is symmetric). Then, there are disjoint open sets  $U_\ell$  and  $U_r$  such that  $\ell \in U_\ell$ ,  $r \in U_r$ , and  $x \prec_f y$  for every  $x \in U_\ell$  and  $y \in U_r$ , see [4]. Next, take  $\ell_n \in U_\ell$  and  $r_m \in U_r$  such that  $n > m$ . Then, we have  $\ell_n \prec_f r_m$ . However, by (3), (4) and (5), we get that  $\{r_m, \ell_n\} \in \langle \{R_m, W_m\} \rangle \cap \mathcal{F}_2(X)$ , and therefore  $f(\{r_m, \ell_n\}) = r_m$ . This is clearly impossible, so  $\ell = r$ .

Having already established this, let us observe that  $b = \ell = r$  implies  $b \in \bar{L} \cap \bar{R}$  because  $\ell \in \bar{L}$  and  $r \in \bar{R}$ . However,  $\bar{L} \cap \bar{R} = \{a\}$  which finally implies that  $b = a$ .

We are now ready to prove that, for instance,  $\{W_n : n < \omega\}$  is a local base at  $a$ . To this end, suppose if possible that this fails. Hence, there exists an open neighbourhood  $U$  of  $a$  such that  $W_n \setminus U \neq \emptyset$  for every  $n < \omega$ . Next, whenever  $n < \omega$ , take a point  $t_n \in W_n \setminus U$ . Since  $X$  is compact,  $\{t_n : n < \omega\}$  has a cluster point  $t \notin U$ . Then,  $t \prec_f a$  and, as before, we may find disjoint open sets  $U_t$  and  $U_a$  such that  $t \in U_t$ ,  $a \in U_a$ , and  $x \prec_f y$  for every  $x \in U_t$  and  $y \in U_a$ . Next, take  $t_n \in U_t$  and  $r_m \in U_a$  such that  $n > m$ . Then,  $t_n \prec_f r_m$ , while, by (4) and (5),  $r_m \prec_f t_n$  because  $\{r_m, t_n\} \in \langle \{R_m, W_m\} \rangle \cap \mathcal{F}_2(X)$ . The contradiction so obtained completes the proof.  $\square$

## 6. PROOF OF THEOREM 1.2

In case  $X$  is a topologically well-orderable space, we may use Theorem 1.1.

Suppose that  $\mathcal{F}_2(X)$  has a  $\tau_F$ -continuous selection. If  $X$  is compact, then Theorem 1.2 is, in fact, a result of van Mill and Wattel [6]. Let  $X$  be non-compact. By Theorem 2.1,  $X$  is locally compact. Then, by Theorem 3.1,

$\mathcal{F}_2(\alpha X)$  has a  $\tau_V$ -continuous selection  $f$  such that  $f^{-1}(\alpha) = \{\{\alpha\}\}$ . Relying once again on the result of [6],  $\alpha X$  is a linear ordered topological space with respect to some linear order “ $<$ ” on  $\alpha X$ . It now suffices to show that there exists a compatible (with the topology of  $\alpha X$ ) linear order “ $\prec$ ” on  $\alpha X$  such that  $\alpha$  is either the first or the last element of  $\alpha X$ , see [2, Lemma 4.1]. We show this following precisely the proof of Theorem 1.1 in [3]. Namely, let

$$L = \{x \in \alpha X : x < \alpha\} \text{ and } R = \{x \in \alpha X : \alpha < x\}.$$

Note that  $L, R \subset \alpha X \setminus \{\alpha\} = X$  are open subsets of  $\alpha X$ . In case one of these sets is also closed, the desired linear order “ $\prec$ ” on  $\alpha X$  can be defined by exchanging the places of  $L$  and  $R$ . Namely, by letting for  $x, y \in \alpha X$  that  $x \prec y$  if and only if

$$\begin{aligned} &x, y \in \bar{L} \text{ and } x < y, \text{ or} \\ &x, y \in \bar{R} \text{ and } x < y, \text{ or} \\ &x \in \bar{R} \text{ and } y \in \bar{L}. \end{aligned}$$

Finally, let us consider the case  $\bar{L} \cap \bar{R} = \{\alpha\}$ . Then,  $\alpha$  is a partition of  $\alpha X$ . Hence, by Theorem 5.1,  $\alpha X$  is first countable at  $\alpha$ . Let  $\mathcal{C}[\alpha]$  be the connected component of  $\alpha$  in  $\alpha X$ . Since  $f^{-1}(\alpha) = \{\{\alpha\}\}$ , it now follows from Theorem 4.1 that  $\mathcal{C}[\alpha] = \{\alpha\}$ . Indeed,  $\mathcal{C}' = \mathcal{C}[\alpha] \cap \{x \in \alpha X : x \leq \alpha\}$  and  $\mathcal{C}'' = \mathcal{C}[\alpha] \cap \{x \in \alpha X : x \geq \alpha\}$  are both connected subsets of  $X$  with  $\alpha \in \mathcal{C}' \cap \overline{(X \setminus \mathcal{C}'})$  and  $\alpha \in \mathcal{C}'' \cap \overline{(X \setminus \mathcal{C}'')}$  (consider that  $X \setminus \mathcal{C}' \supset R$  and  $X \setminus \mathcal{C}'' \supset L$ ), so that  $\mathcal{C}' = \{\alpha\}$  and  $\mathcal{C}'' = \{\alpha\}$ , whence also  $\mathcal{C}[\alpha] = \{\alpha\}$ . Then,  $\alpha X$  has a clopen base at  $\alpha$ . Indeed, let  $\ell \in L$  and  $r \in R$ . Since  $\mathcal{C}[\alpha]$  is also the quasi-component of the point  $\alpha$ , there are clopen neighbourhoods  $U_\ell, U_r$  of  $\alpha$  such that  $\ell \notin U_\ell$  and  $r \notin U_r$ . Then,

$$U = \{x \in U_\ell \cap U_r : \ell < x < r\} = \{x \in U_\ell \cap U_r : \ell \leq x \leq r\}$$

is a clopen neighbourhood of  $\alpha$  with  $U \subset \{x \in X : \ell < x < r\}$ .

That is,  $\alpha X$  has a clopen base at  $\alpha$  and it is first countable at this point. Then, let  $\{U_n : n < \omega\}$  be a decreasing clopen base at  $\alpha$ , with  $U_0 = \alpha X$ . Next, for every point  $x \in X$ , let  $n(x) = \max\{n : x \in U_n\}$  and, for convenience,  $n(\alpha) = \omega$ . Finally, we may define a linear order “ $\prec$ ” on  $\alpha X$  by putting  $x \prec y$  if and only if

$$\text{either } n(x) < n(y) \text{ or } n(x) = n(y) \text{ and } x < y.$$

Since  $\{U_n : n < \omega\}$  is a decreasing clopen base at  $\alpha$ , the order “ $\prec$ ” is compatible with the topology of  $\alpha X$ . It is clear that, with respect to “ $\prec$ ”,  $\alpha$  is the last element of  $X$ . This completes the proof.

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