FELL CONTINUOUS SELECTIONS AND TOPOLOGICALLY WELL-ORDERABLE SPACES

V. GUTEV AND T. NOGURA

School of Mathematical Sciences, Faculty of Science, University of
KwaZulu-Natal, King George V Avenue, Durban 4041, South Africa
E-mail address: gutev@ukzn.ac.za
URL: maths.za.org

Department of Mathematics, Faculty of Science, Ehime University, Matsuyama,
790-8577 Japan
E-mail address: nogura@dpc.ehime-u.ac.jp

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Abstract. The present paper extends the idea to characterize topological properties of a space X by means of continuous selections for its closed subsets $\mathcal{F}(X)$ endowed with a "natural" hyperspace topology. In this particular case, we prove that the property of X to be topologically well-orderable is equivalent to the existence of a selection for $\mathcal{F}(X)$ which is continuous with respect to the Fell topology.

1. Introduction

Let X be a topological space, and let $\mathcal{F}(X)$ be for the family of the non-empty closed subsets of X. Also let τ be a topology on $\mathcal{F}(X)$. A map $f : \mathcal{F}(X) \to X$ is a *selection* for $\mathcal{F}(X)$ if $f(S) \in S$ for every $S \in \mathcal{F}(X)$. A map $f : \mathcal{F}(X) \to X$ is a τ*-continuous* selection for $\mathcal{F}(X)$ if it is a selection for $\mathcal{F}(X)$ which is continuous with respect to the topology τ on $\mathcal{F}(X)$.

Now, we recall two topologies which will play the most important role in this paper. The one is the *Vietoris topology* τ_V on $\mathcal{F}(X)$ which is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \cap V \neq \emptyset,\ V \in \mathcal{V},\ \text{and}\ S \subset \bigcup \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of X. The another is the *Fell topology* τ_F which is defined by all basic Vietoris neighbourhoods $\langle \mathcal{V} \rangle$ with the property that $X \setminus \bigcup \mathcal{V}$ is compact.

Finally, let us recall that a space X is called *topologically well-orderable* (see Engelking, Heath and Michael [1]) if there exists a linear order on X compatible with the topology of X such that every non-empty closed subset of X has a first element.

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On the one hand, Engelking, Heath and Michael have characterized in [1, Theorem 4.4] all topologically well-orderable spaces among the strongly zero-dimensional metrizable spaces (see, also, Herrlich [3]). Here is their result.

Theorem 1.1 ([1]). *A strongly zero-dimensional metrizable space is topologically well-orderable if and only if it is locally compact and separable.*

On the other hand, just in this case, we also have the following recent result of ours [2, Theorem 6.1].

Theorem 1.2 ([2]). *A strongly zero-dimensional metrizable space X admits a τ_F-continuous selection for $\mathcal{F}(X)$ if and only if it is locally compact and separable.*

In the present note we prove the following natural result which generalizes both Theorem 1.1 and Theorem 1.2.

Theorem 1.3. *A Hausdorff space X is topologically well-orderable if and only if $\mathcal{F}(X)$ has a τ_F-continuous selection.*

The proof of Theorem 1.3 is a based on a reduction to the compact case and special Vietoris continuous selections. Here are the precise steps of this proof.

2. A reduction to locally compact spaces

In the sequel, all spaces are assumed to be at least Hausdorff.

Theorem 2.1. *Let X be a space such that $\mathcal{F}(X)$ has a τ_F-continuous selection. Then X is locally compact.*

Proof. Let $f : \mathcal{F}(X) \to X$ be a τ_F-continuous selection for $\mathcal{F}(X)$. Suppose, if possible, that X is not locally compact. Hence, there exists a point $p \in X$ such that $\overline{V}$ is not compact for every neighbourhood V of p in X. Claim that there exists $S \in \mathcal{F}(X)$ such that

$$p \notin S \quad \text{and} \quad f(S \cup \{p\}) = p. \tag{1}$$

For the purpose, note that there exists $F \in \mathcal{F}(X)$ such that F is not compact and $p \notin F$. Next, set $Y = F \cup \{p\}$ and $g = f|\mathcal{F}(Y)$. Thus, we get a τ_F-continuous selection for $\mathcal{F}(Y)$ because Y is closed in X. Since p is an isolated point of Y, we now have that $g^{-1}(p)$ is a τ_F-clopen neighbourhood of $\{p\}$ in $\mathcal{F}(Y)$. Hence, there exists a finite family $\mathcal{W}$ of open subsets of Y such that $Y \setminus \bigcup \mathcal{W}$ is compact and

$$\{p\} \in \langle \mathcal{W} \rangle \subset g^{-1}(p).$$

Then, $F \cap (\bigcup \mathcal{W}) \neq \emptyset$ because F is not compact. Therefore, there exists a non-empty finite subset S of F such that $S \cup \{p\} \in \langle \mathcal{W} \rangle$. This S is as required.

Let S be as in (1). Since X is Hausdorff, $f(S) \neq f(S \cup \{p\})$, and f is τ_F-continuous, there exist two finite families $\mathcal{U}$ and $\mathcal{V}$ of open subsets of X such that

$X\backslash\bigcup\mathcal{U}$ and $X\backslash\bigcup\mathcal{V}$ are compact, $S\in\langle\mathcal{U}\rangle$, $S\cup\{p\}\in\langle\mathcal{V}\rangle$, and $\langle\mathcal{U}\rangle\cap\langle\mathcal{V}\rangle=\emptyset$. Then, the set $\mathcal{V}_p=\{V\in\mathcal{V}:V\cap S=\emptyset\}$ is such that

$$p\in\bigcap\mathcal{V}_p\subset X\backslash\bigcup\mathcal{U}. \tag{2}$$

Indeed, $\mathcal{V}_p\neq\emptyset$ because $S\notin\langle\mathcal{V}\rangle$. Hence, $p\in\bigcap\mathcal{V}_p$. Suppose there is a point $q\in(\bigcap\mathcal{V}_p)\cap(\bigcup\mathcal{U})$. Then, $S\cup\{q\}\in\langle\mathcal{U}\rangle$ because $S\in\langle\mathcal{U}\rangle$. However, we also get that $S\cup\{q\}\in\langle\mathcal{V}\rangle$ because $S\cup\{p\}\in\langle\mathcal{V}\rangle$, while $S\cap V=\emptyset$ for some $V\in\mathcal{V}$ implies $q\in V$. Thus, we finally get that $S\cup\{q\}\in\langle\mathcal{U}\rangle\cap\langle\mathcal{V}\rangle$ which is impossible. So, (2) holds.

To finish the proof, it only remains to observe that the statement (2) contradicts the choice of p. Namely $V_p=\bigcap\mathcal{V}_p$ becomes a neighbourhood of p which, by (2), has a compact closure $\overline{V_p}$ because $X\backslash\bigcup\mathcal{U}$ is compact. □

3. A reduction to compact spaces

For a locally compact space X we will use αX to denote the one point compactification of X. For a non-compact locally compact X let us agree to denote by α the point of the singleton $\alpha X\backslash X$.

Theorem 3.1. *Let X be a locally compact non-compact space. Then, $\mathcal{F}(X)$ admits a τ_F-continuous selection if and only if $\mathcal{F}(\alpha X)$ admits a τ_V-continuous selection g such that $g^{-1}(\alpha)=\{\{\alpha\}\}$.*

Proof. Let $f:\mathcal{F}(X)\to X$ be a τ_F-continuous selection for $\mathcal{F}(X)$. Define a selection $g:\mathcal{F}(\alpha X)\to\alpha X$ by $g(S)=f(S\cap X)$ if $S\cap X\neq\emptyset$ and $g(S)=\alpha$ otherwise. To show that g is as required, we have only to show that it is τ_V-continuous. To this end, note that any selection is τ_V-continuous at the singletons of $\mathcal{F}(\alpha X)$. Hence, in particular, g is τ_V-continuous at $\{\alpha\}$. Take now $S\in\mathcal{F}(\alpha X)$ such that $S\cap X\neq\emptyset$, and let U be a neighbourhood of $g(S)=f(S\cap X)$. Since f is τ_F-continuous, there exists a finite family $\mathcal{V}$ of open subsets of X such that $X\backslash\bigcup\mathcal{V}$ is compact, $S\cap X\in\langle\mathcal{V}\rangle$, and

$$f(\langle\mathcal{V}\rangle)\subset U. \tag{3}$$

Then, by the definition of the topology at the point α, we get that

$$V_0=\left(\bigcup\mathcal{V}\right)\cup\{\alpha\} \tag{4}$$

is a neighbourhood of α in αX because $X\backslash\bigcup\mathcal{V}$ is compact. Finally, set

$$\mathcal{V}_0=\{V_0\}\cup\mathcal{V} \tag{5}$$

In this way, we get a finite family $\mathcal{V}_0$ of open subset of αX because X is open in αX. Take $T\in\mathcal{F}(\alpha X)$ and then note that $T\in\langle\mathcal{V}_0\rangle$ implies that $T\cap X\cap V\neq\emptyset$ for every $V\in\mathcal{V}$. Hence, by (4) and (5), we get that

$$T\in\langle\mathcal{V}_0\rangle\quad\text{if and only if}\quad T\cap X\in\langle\mathcal{V}\rangle.$$

In particular, by (3), this implies that $\langle \mathcal{V}_0 \rangle$ is a τ_V-neighbourhood of S in $\mathcal{F}(\alpha X)$ such that $g(\langle \mathcal{V}_0 \rangle) \subset U$. That is, g is τ_V-continuous at S.

Suppose now that $g : \mathcal{F}(\alpha X) \to \alpha X$ is a τ_V-continuous selection for $\mathcal{F}(\alpha X)$ such that $g^{-1}(\alpha) = \{\{\alpha\}\}$. Note that $S \cup \{\alpha\} \in \mathcal{F}(\alpha X)$ whenever $S \in \mathcal{F}(X)$. Then, we may define a selection $f : \mathcal{F}(X) \to X$ for $\mathcal{F}(X)$ by

$$f(S) = g\left(S \cup \{\alpha\}\right) \quad \text{for every } S \in \mathcal{F}(X).$$

Let us show that f is τ_F-continuous. Take $S \in \mathcal{F}(X)$ and a neighbourhood U of $f(S) = g(S \cup \{\alpha\})$. Then, there exists a finite family $\mathcal{W}$ of open subsets of αX such that $S \cup \{\alpha\} \in \langle \mathcal{W} \rangle$ and $g(\langle \mathcal{W} \rangle) \subset U$. Then,

$$\mathcal{V} = \{W \backslash \{\alpha\} : W \in \mathcal{W}\}$$

is a finite family of open subsets of X such that, for every $T \in \mathcal{F}(X)$,

$$T \in \langle \mathcal{V} \rangle \quad \text{if and only if} \quad T \cup \{\alpha\} \in \langle \mathcal{W} \rangle.$$

Hence, in particular, $S \in \langle \mathcal{V} \rangle$ and $g(\langle \mathcal{V} \rangle) \subset U$. In addition, $X \backslash \bigcup \mathcal{V}$ is compact because $X \backslash \bigcup \mathcal{V} = \alpha X \backslash \bigcup \mathcal{W}$. That is, g is τ_F-continuous. □

4. Special selections and connected sets

As a next step in our proof, we establish the following result which may have some independent interest.

Theorem 4.1. *Let X be a space, $a \in X$, and let $A \in \mathcal{F}(X)$ be a connected set such that $|A| > 1$ and $\{a\} = A \cap \overline{X \backslash A}$. Also, suppose that $f : \mathcal{F}(X) \to X$ is a τ_V-continuous selection for $\mathcal{F}(X)$. Then, $f^{-1}(a) \neq \{\{a\}\}$.*

Proof. Suppose, if possible, that $f^{-1}(a) = \{\{a\}\}$. Then, $f(A) \neq \{a\}$ because, by hypothesis, $|A| > 1$. Therefore, $U = X \backslash \overline{X \backslash A}$ is a neighbourhood of $f(A)$. Since f is τ_V-continuous, this implies the existence of a finite open (in X) cover $\mathcal{V}$ of A such that

$$f(\langle \mathcal{V} \rangle) \subset U. \tag{6}$$

By hypothesis, $\{a\} = A \cap \overline{X \backslash A}$. Hence, every neighbourhood of A contains a point of $X \backslash A$. Therefore, there exists a non-empty finite set $F \subset X \backslash A$ with

$$Y = A \cup F \in \langle \mathcal{V} \rangle. \tag{7}$$

Define a map $\varphi : \mathcal{F}(A) \to \mathcal{F}(Y)$ by $\varphi(S) = S \cup F$ for every $S \in \mathcal{F}(A)$. Note that φ is continuous with respect to the Vietoris topologies on $\mathcal{F}(A)$ and $\mathcal{F}(Y)$. On the other hand, by a result of [4, 6], the space $(\mathcal{F}(A), \tau_V)$ is connected because so is A. Hence, $\varphi(\mathcal{F}(A))$ is a connected subset of $(\mathcal{F}(Y), \tau_V)$. Finally, observe that

$$\varphi(\mathcal{F}(A)) \cap f^{-1}(F) \neq \emptyset. \tag{8}$$

Indeed, by assumption, we have $f^{-1}(a) = \{\{a\}\}$ which implies $f(\{a\} \cup F) \in F$. Therefore $\{a\} \cup F \in f^{-1}(F) \cap \varphi(\mathcal{F}(A))$.

To finish the proof, let us observe that the set $f^{-1}(F)$ is τ_V-clopen in $\mathcal{F}(Y)$ because F is clopen in Y. Hence, by (8), $\varphi(\mathcal{F}(A)) \subset f^{-1}(F)$ because $\varphi(\mathcal{F}(A))$ is τ_V-connected. In particular, this implies that $f(A \cup F) \in F \subset X \setminus U$. However, by (6) and (7), we have $f(A \cup F) \in U$. A contradiction. □

5. A further result about special selections

In what follows, we shall say that a point $a \in X$ is a *partition* of X if there are open subset $L, R \subset X \setminus \{a\}$ such that $\overline{L} \cap \overline{R} = \{a\}$ and $L \cap R = \emptyset$.

We complete the preparation for the proof of Theorem 1.3 with the following result about special Vietoris continuous selections and partitions.

Theorem 5.1. *Let X be a compact space, f be a τ_V-continuous selection for $\mathcal{F}(X)$, and let $a \in X$ be a partition of X, with $f^{-1}(a) = \{\{a\}\}$. Then, X is first countable at a.*

Proof. By definition, there are open sets $L, R \subset X \setminus \{a\}$ such that $\overline{L} \cap \overline{R} = \{a\}$ and $L \cap R = \emptyset$. Hence, both L and R are non-empty. Take a point $\ell_1 \in L$. Then, by hypothesis, $f(\{\ell_1, a\}) = \ell_1$. Since f is τ_V-continuous, this implies the existence of a neighbourhood $L_1 \subset L$ of ℓ_1 and a neighbourhood V_1 of a such that

$$L_1 \cap V_1 = \emptyset \qquad \text{and} \qquad f(\langle\{L_1, V_1\}\rangle) \subset L_1.$$

Since $a \in \overline{R}$, there now exists a point $r_1 \in V_1 \cap R$. On the other hand, $f(\{a, r_1\}) = r_1 \in V_1$. Hence, just like before, there exists a neighbourhood $R_1 \subset V_1 \cap R$ of r_1 and a neighbourhood W_1 of a such that

$$\overline{W_1} \subset V_1, \quad R_1 \cap W_1 = \emptyset \qquad \text{and} \qquad f(\langle\{R_1, W_1\}\rangle) \subset R_1.$$

Thus, by induction, we may construct a sequence $\{\ell_n : n < \omega\}$ of points of L, a sequence $\{r_n : n < \omega\}$ of points of R, and open sets $L_n, V_n, R_n, W_n \subset X$ such that

$$\ell_n \in L_n, \quad a \in V_n, \quad L_n \cap V_n = \emptyset \quad \text{and} \quad f(\langle\{L_n, V_n\}\rangle) \subset L_n, \tag{9}$$

$$r_n \in R_n, \quad a \in W_n, \quad R_n \cap W_n = \emptyset \quad \text{and} \quad f(\langle\{R_n, W_n\}\rangle) \subset R_n, \tag{10}$$

and

$$\overline{V_{n+1}} \subset W_n \subset \overline{W_n} \subset V_n, \quad L_{n+1} \subset L \cap W_n \quad \text{and} \quad R_n \subset R \cap V_n. \tag{11}$$

Let $C = \bigcap\{W_n : n < \omega\}$. Note that $C \subset X$ is closed because, by (11),

$$C = \bigcap\left\{\overline{W_n} : n < \omega\right\} = \bigcap\left\{\overline{V_n} : n < \omega\right\}.$$

Let us show that, in fact, $C = \{a\}$. To this end, for every $n < \omega$ we set $P_n = \overline{W_n} \cup \{\ell_n\}$. Thus, we get a decreasing sequence of closed subsets because, by (9) and (11), $P_{n+1} = \overline{W_{n+1}} \cup \{\ell_{n+1}\} \subset W_n \subset P_n$. The last observation also implies that $C = \bigcap\{P_n : n < \omega\}$. Since X is compact, we finally get that P_n is τ_V-convergent to C, i.e. $f(C) = \lim_{n\to\infty} P_n$. On the other hand, by (9) and (11), we get that $P_n \in \langle\{V_n, L_n\}\rangle$, hence $f(P_n) = \ell_n \in L$. As a result, $f(C) \in \overline{L}$.

We repeat the same trick with R. Namely, for every $n < \omega$ we set $Q_n = \overline{V_{n+1}} \cup \{r_n\}$. Then, by (10) and (11),

$$Q_{n+1} = \overline{V_{n+2}} \cup \{r_{n+1}\} \subset V_{n+1} \subset Q_n \in \langle\{R_n, W_n\}\rangle.$$

Since X is compact, we now have $f(C) = \lim_{n\to\infty} Q_n \in \overline{R}$ because, by (10), $f(Q_n) = r_n \in R$. However, by hypothesis, $\overline{L} \cap \overline{R} = \{a\}$ which finally implies that $C = \{a\}$ because $C \in f^{-1}(a) = \{\{a\}\}$.

Thus, we have that, for instance, $\{W_n : n < \omega\}$ is a local base at a because X is compact and $\overline{W_{n+1}} \subset W_n$ for every n. □

6. Proof of Theorem 1.3

In case $(X, \prec)$ is a topologically well-ordered space, $f(S) = \min_\prec S$, $S \in \mathcal{F}(X)$, defines a selection f for $\mathcal{F}(X)$. Since, by [1, Lemma 4.1], the set $\{x \in X : x \preceq a\}$ is compact for every $a \in X$, the map f is τ_F-continuous.

Suppose now that $\mathcal{F}(X)$ has a τ_F-continuous selection. In case X is compact, the conclusion of Theorem 1.3 follows from a result of van Mill and Wattel [5]. Let X be non-compact. By Theorem 2.1, X is locally compact. Then, by Theorem 3.1, $\mathcal{F}(\alpha X)$ has a τ_V-continuous selection f such that $f^{-1}(\alpha) = \{\{\alpha\}\}$. Relying once again to the result of [5], αX is a linear ordered topological space with respect to some linear order $<$ on αX. It now suffices to show that there exists a compatible (with the topology of αX) linear order $\prec$ on αX such that α is either the first or the last element of αX. For the purpose, let

$$L = \{x \in \alpha X : x < \alpha\} \quad \text{and} \quad R = \{x \in \alpha X : \alpha < x\}.$$

Note that $L, R \subset \alpha X \setminus \{\alpha\} = X$ are open. In case one of these sets is also closed, the desired linear order $\prec$ on αX can be defined by exchanging the places of L and R. Namely, by letting for $x, y \in \alpha X$ that $x \prec y$ if and only if

$$x, y \in \overline{L} \text{ and } x < y, \text{ or } x, y \in \overline{R} \text{ and } x < y, \text{ or } x \in \overline{R} \text{ and } y \in \overline{L}.$$

Finally, let us consider the case $\overline{L} \cap \overline{R} = \{\alpha\}$. In this case, α is a partition of αX. Hence, by Theorem 5.1, αX is first countable at α. Let $\mathcal{C}[\alpha]$ be the connected component of αX in α. Since $f^{-1}(\alpha) = \{\{\alpha\}\}$, it now follows from Theorem 4.1, that $\mathcal{C}[\alpha] = \{\alpha\}$. Then, αX has a clopen base at α. Indeed, let $\ell \in L$ and

$r \in R$. Since $\mathcal{C}[\alpha]$ is also the quasi-component of the point α, there are clopen neighbourhoods U_ℓ, U_r of α such that $\ell \notin U_\ell$ and $r \notin U_r$. Then,

$$U = \{x \in U_\ell \cap U_r : \ell < x < r\} = U_\ell \cap U_r \cap (\ell, r)$$

is a clopen neighbourhood of α, with $U \subset (\ell, r)$.

That is, αX has a clopen base at α and it is first countable at this point. Then, let $\{U_n : n < \omega\}$ be a decreasing clopen base at α with $U_0 = \alpha X$. Next, for every point $x \in X$ let $n(x) = \max\{n : x \in U_n\}$ and, for convenience, $n(\alpha) = \omega$. Then, we may define a linear order $\prec$ on αX by $x \prec y$ if and only if

$$\text{either} \quad n(x) < n(y) \quad \text{or} \quad n(x) = n(y) \text{ and } x < y.$$

Since $\{U_n : n < \omega\}$ is a decreasing clopen base at α, the order $\prec$ so defined is compatible with the topology of αX. It is clear that, with respect to $\prec$, α is the last element of $(X, \prec)$. This completes the proof.

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School of Mathematical Sciences, Faculty of Science, University of KwaZulu-Natal, King George V Avenue, Durban 4041, South Africa
E-mail address: gutev@ukzn.ac.za
URL: maths.za.org

Department of Mathematics, Faculty of Science, Ehime University, Matsuyama, 790-8577 Japan
E-mail address: nogura@dpc.ehime-u.ac.jp