A TOPOLOGY GENERATED BY SELECTIONS

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Topology Appl., in print

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Abstract. Every selection $f : \mathcal{F}_2(X) \to X$ for the family $\mathcal{F}_2(X)$ of at most two-point subsets of a set X naturally defines an order-like relation $\preceq_f$ on X by letting $x \preceq_f y$ if and only if $f(\{x, y\}) = x$. In particular, it also defines a natural interval-like topology on X. In the present paper we study properties of this topology, and its possible relations with other topologies on X with respect to which f is Vietoris continuous.

1. Introduction

Let $(X, \mathcal{T})$ be a T_1-space, where $\mathcal{T}$ is the topology of X, and let $\mathcal{F}(X, \mathcal{T})$ be the family of all non-empty closed subsets of $(X, \mathcal{T})$. Also, for every $n \geq 1$, let $\mathcal{F}_n(X) = \{S \subset X : 0 < |S| \leq n\}$. Note that $\mathcal{F}_n(X) \subset \mathcal{F}(X, \mathcal{T})$ because $(X, \mathcal{T})$ is a T_1-space. Hence, we may consider $\mathcal{F}(X, \mathcal{T})$ as an extension of X identifying X with $\mathcal{F}_1(X)$. From this point of view, a topology τ on $\mathcal{F}(X, \mathcal{T})$ is *admissible* [5] if its restriction on X coincides with $\mathcal{T}$.

Let $\mathcal{D} \subset \mathcal{F}(X, \mathcal{T})$, and let τ be an admissible topology on $\mathcal{F}(X, \mathcal{T})$. A map $f : \mathcal{D} \to X$ is a *selection* for $\mathcal{D}$ if $f(S) \in S$ for every $S \in \mathcal{D}$. A map $f : \mathcal{D} \to X$ is a *τ-continuous* selection for $\mathcal{D}$ if it is a selection for $\mathcal{D}$ which is continuous with respect to the relative topology on $\mathcal{D}$ as a subspace of $(\mathcal{F}(X, \mathcal{T}), \tau)$.

So far one of the best admissible topologies on $\mathcal{F}(X, \mathcal{T})$ is the Vietoris one $\tau_{V(\mathcal{T})}$. Let us recall that $\tau_{V(\mathcal{T})}$ is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X, \mathcal{T}) : S \cap V \neq \emptyset,\ V \in \mathcal{V},\ \text{ and } S \subset \bigcup \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of $(X, \mathcal{T})$.

Finally, let $\mathcal{S}e\ell(X, \mathcal{T})$ be the set of all $\tau_{V(\mathcal{T})}$-continuous selections for $\mathcal{F}(X, \mathcal{T})$, and $\mathcal{S}e\ell_2(X, \mathcal{T})$ that of all $\tau_{V(\mathcal{T})}$-continuous selections for $\mathcal{F}_2(X)$. Also, we will use $\mathcal{S}e\ell_2(X)$ to denote the set of all selections for $\mathcal{F}_2(X)$. Note that always $\mathcal{S}e\ell_2(X, \mathcal{T}) \subset \mathcal{S}e\ell_2(X)$, while $\mathcal{S}e\ell_2(X, \mathcal{T}) \neq \emptyset$ provided $\mathcal{S}e\ell(X, \mathcal{T}) \neq \emptyset$.

Date: June 17, 2003.

1991 *Mathematics Subject Classification.* 54B20, 54C65.

Key words and phrases. Hyperspace topology, Vietoris topology, continuous selection.

Research of the first author is supported in part by the National Research Foundation under Grant number 2053735.

Every selection $f \in \mathcal{S}e\ell_2(X)$ defines an order-like relation $\preceq_f$ on X (see Michael [5]) by letting for $x, y \in X$ that

$$x \preceq_f y \qquad \text{if and only if} \qquad f(\{x, y\}) = x.$$

In the sequel, we shall regard $\preceq_f$ as an f-*order* on X. Also, we shall write that $x \prec_f y$ provided $x \preceq_f y$ and $x \neq y$.

Now, to every selection $f \in \mathcal{S}e\ell_2(X)$ we may associate a topology $\mathcal{T}_f$ on X generated by all "open f-intervals" $\{y \in x : y \prec_f x\}$ and $\{y \in X : x \prec_f y\}$, $x \in X$. Some properties of this topology were investigated in [3]. For instance, $\mathcal{T}_f$ is always a Hausdorff topology ([3, Lemma 2.3]), while $\mathcal{T}_f \subset \mathcal{T}$ whenever $(X, \mathcal{T})$ is a Hausdorff space and $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ ([3, Theorem 3.5]). On the other hand, f may fail to be Vietoris continuous with respect to $\mathcal{T}_f$, see [3, Example 3.6].

In the present paper we extend our efforts in this direction. First, we generalize our previous result demonstrating that $\mathcal{T}_f$ is, in fact, always a regular topology on X, see Corollary 2.3. As a result, we also demonstrate that, for every $f \in \mathcal{S}e\ell_2(\mathbb{Q}, \mathcal{T}_e)$, where $\mathcal{T}_e$ is the Euclidean topology on the rational numbers $\mathbb{Q}$, the topology $\mathcal{T}_f$ is again the Euclidean topology on $\mathbb{Q}$ (Corollary 2.4) but, in general, $\mathcal{T}_f \neq \mathcal{T}_e$ (Corollary 3.4). The latter is related to a special construction of selections (see Theorem 3.1) which may have an independent interest. Another interesting construction of selections $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$, with $f \notin \mathcal{S}e\ell_2(X, \mathcal{T}_f)$, is achieved as well, see Theorem 4.1.

2. A topology generated by selections

Let X be a set, and let $E \subset X \times X$ be a *relation* on X. As usual, we write xEy to denote that $(x, y) \in E$. Now, let us recall that a relation E on X is *anti-symmetric* if xEy and yEx implies $x = y$. We shall say that an anti-symmetric relation E on X is a *selection relation* if xEy or yEx for every $(x, y) \in X^2$. Finally, let us stress the attention that, in this terminology, a relation E on X is a *linear order* on X if E is a selection relation which is also transitive (i.e., xEy and yEz implies xEz).

It is clear that the set $\mathcal{S}e\ell_2(X)$ corresponds to all possible selection relations on X. Namely, any selection relation E on X defines a selection $f_E : \mathcal{F}_2(X) \to X$ by letting $f_E(\{x, y\}) = x$ iff xEy. On the other hand, if $f : \mathcal{F}_2(X) \to X$ is a selection, then $\preceq_f$ is a selection relation on X.

Suppose now that $\preceq$ is a selection relation on X. As before, for convenience, we shall write that $x \prec y$ to denote that $x \preceq y$ and $x \neq y$. Then, whenever $x \in X$, we may consider the following $\preceq$-*intervals* generated by $\preceq$:

$$(\leftarrow, x]_{\preceq} = \{z \in X : z \preceq x\} \qquad \text{and} \qquad [x, \rightarrow)_{\preceq} = \{z \in X : x \preceq z\},$$
$$(\leftarrow, x)_{\preceq} = \{z \in X : z \prec x\} \qquad \text{and} \qquad (x, \rightarrow)_{\preceq} = \{z \in X : x \prec z\}.$$

Once we have some subsets of X generated by a selection relation $\preceq$, we may associate also a topology $\mathcal{T}_\preceq$ on X. Namely, let $\mathcal{T}_\preceq$ denote the topology on X generated by all "open" $\preceq$-intervals $(\leftarrow, x)_\preceq$ and $(x, \rightarrow)_\preceq$, $x \in X$. Let us explicitly stress the attention that $(X, \mathcal{T}_\preceq)$ is called *orderable* if $\preceq$ is a linear order on X. Unfortunately, the selection relation $\preceq$ may fail to be transitive. In fact, the clopen subsets of $\mathcal{T}_\preceq$ provide an indication about the non-transitivity of $\preceq$. The following very simple observation was established in [3, Proposition 2.2].

Proposition 2.1 ([3]). *Let X be a set, $\preceq$ be a selection relation on X, and let $x, y, z \in X$ be such that $z \prec x \prec y \prec z$. Then, $(\leftarrow, x]_\preceq \cap [y, \rightarrow)_\preceq$ is a clopen subset of $(X, \mathcal{T}_\preceq)$ which separates z from the two-point set $\{x, y\}$.*

Another important property, established in [3, Lemma 2.3], is that $\mathcal{T}_\preceq$ is always a Hausdorff topology on X. Following the arguments in [2, Theorem 3.1], we are going to generalize this result by showing that $\mathcal{T}_\preceq$ is always a regular topology. Towards this end, for a subset $B \subset X$, we let $\overline{B}^\preceq$ to be the closure of B with respect to $\mathcal{T}_\preceq$.

Lemma 2.2. *Let X be a set, $\preceq$ be a selection relation on X, and let $x, z \in X$ be such that $z \prec x$. Then, there is $U \in \mathcal{T}_\preceq$, with $z \in U$ and $\overline{U \cap (\leftarrow, x)_\preceq}^\preceq \subset (\leftarrow, x)_\preceq$. In particular, $x \in V = X \setminus \overline{U \cap (\leftarrow, x)_\preceq}^\preceq$ and $\overline{V \cap (z, \rightarrow)_\preceq}^\preceq \subset (z, \rightarrow)_\preceq$.*

Proof. Let $(z, x)_\preceq = \{y \in X : z \prec y \prec x\}$. We have the following two possibilities. In case there exists a point $y \in (z, x)_\preceq$, we can take $U = (\leftarrow, y)_\preceq$. Then, clearly $U \in \mathcal{T}_\preceq$, and $z \in U$. On the other hand, both $(\leftarrow, y]_\preceq$ and $(\leftarrow, x]_\preceq$ are $\mathcal{T}_\preceq$-closed, while $x \notin (\leftarrow, y]_\preceq$. Since $(\leftarrow, y)_\preceq \cap (\leftarrow, x)_\preceq \subset (\leftarrow, y]_\preceq \cap (\leftarrow, x]_\preceq$, we have that

$$\overline{U \cap (\leftarrow, x)_\preceq}^\preceq \subset (\leftarrow, y]_\preceq \cap (\leftarrow, x]_\preceq \subset (\leftarrow, x)_\preceq.$$

Suppose now that $(z, x)_\preceq = \emptyset$. Then, $y \prec x$ will imply that $y \preceq z$. Hence, we have $(\leftarrow, x)_\preceq \subset (\leftarrow, z]_\preceq$, and therefore $(\leftarrow, z]_\preceq = (\leftarrow, x)_\preceq \cup (\leftarrow, z)_\preceq \in \mathcal{T}_\preceq$. Thus, $(\leftarrow, z]_\preceq$ is a $\mathcal{T}_\preceq$-clopen neighbourhood of z, which does not contain x, so we can take $U = (\leftarrow, z]_\preceq$. □

According to the definition of $\mathcal{T}_\preceq$, we have the following immediate consequence.

Corollary 2.3. *Let X be a set, and let $\preceq$ be a selection relation on X. Then, the topology $\mathcal{T}_\preceq$ is regular.*

It will be interesting to see if $\mathcal{T}_\preceq$ is always a normal topology. A possible approach is to follow the case when $\preceq$ is a linear order on X, and to check if the resulting topological space $(X, \mathcal{T}_\preceq)$ is *collectionwise Hausdorff*. Thus, we have the following natural question.

▷ **Question 1.** Let $\preceq$ be a selection relation on a set X, and let $Z \subset X$ be a $\mathcal{T}_{\preceq}$-discrete set of points of X. Does there exist a disjoint family $\{U_z : z \in Z\} \subset \mathcal{T}_{\preceq}$ such that $z \in U_z$ for every $z \in Z$?

To prepare for our next result, let us recall that, for a selection $f : \mathcal{F}_2(X) \to X$, we merely let $\mathcal{T}_f = \mathcal{T}_{\preceq_f}$, where $\preceq_f$ is the selection relation generated by f.

Corollary 2.4. *Let $\mathbb{Q}$ be the set of the rational numbers, $\mathcal{T}_e$ be the Euclidean topology on $\mathbb{Q}$, and let $f \in \mathcal{S}e\ell_2(\mathbb{Q}, \mathcal{T}_e)$. Then, $\mathcal{T}_f \subset \mathcal{T}_e$ and $(\mathbb{Q}, \mathcal{T}_f)$ is homeomorphic to $(\mathbb{Q}, \mathcal{T}_e)$.*

Proof. Since $(\mathbb{Q}, \mathcal{T}_e)$ is a Hausdorff space, by [3, Theorem 3.5], $\mathcal{T}_f \subset \mathcal{T}_e$. By Corollary 2.3, $\mathcal{T}_f$ is regular, while it is clear that $\mathcal{T}_f$ is second countable. Hence, according to the Urysohn's metrization theorem [8], $(\mathbb{Q}, \mathcal{T}_f)$ is metrizable. However, $\mathbb{Q}$ is countable, and $(\mathbb{Q}, \mathcal{T}_f)$ is dense in itself because $\mathcal{T}_f \subset \mathcal{T}_e$. Therefore, by a result of [7] (see, also, [1, 6.2.A(d)]), $(\mathbb{Q}, \mathcal{T}_f)$ is homeomorphic to $(\mathbb{Q}, \mathcal{T}_e)$. □

According to Corollary 2.4, the identity map $1_{\mathbb{Q}} : (\mathbb{Q}, \mathcal{T}_e) \to (\mathbb{Q}, \mathcal{T}_f)$ is bijective and continuous, but, in general, it is not a homeomorphism. In fact, $1_{\mathbb{Q}}$ will be a homeomorphism if and only if $\mathcal{T}_f = \mathcal{T}_e$. As we will see in the next section, this is not true for a natural class of spaces, in particular for $(\mathbb{Q}, \mathcal{T}_e)$ as well.

Another interesting point is to see if Corollary 2.4 can be extended to other spaces. Looking at this direction, the best is probably to consider first the space of the irrational numbers which gives rise the following natural question.

▷ **Question 2.** Let $\mathbb{P}$ be the set of the irrational numbers, $\mathcal{T}_e$ be the Euclidean topology on $\mathbb{P}$, and let $f \in \mathcal{S}e\ell_2(\mathbb{P}, \mathcal{T}_e)$. Is it true that $(\mathbb{P}, \mathcal{T}_f)$ is again the space of the irrational numbers?

However, we don't know even if the resulting selection topology $\mathcal{T}_f$ on $\mathbb{P}$ will be first countable.

▷ **Question 3.** Let $\mathbb{P}$ be the irrational numbers, and let $\mathcal{T}_e$ be the Euclidean topology on $\mathbb{P}$. Then, does there exist a selection $f \in \mathcal{S}e\ell_2(\mathbb{P}, \mathcal{T}_e)$ such that $\mathcal{T}_f$ is not first countable?

Concerning Question 3, let us observe that, for every $f \in \mathcal{S}e\ell_2(\mathbb{P}, \mathcal{T}_e)$, the corresponding space $(\mathbb{P}, \mathcal{T}_f)$ contains a dense subset which is homeomorphic to the space of the rational numbers $(\mathbb{Q}, \mathcal{T}_e)$. Namely, take a point $p \in \mathbb{P}$, and then consider the copy $Q = \{p + q : q \in \mathbb{Q}\}$ of $\mathbb{Q}$ which is dense in $(\mathbb{P}, \mathcal{T}_e)$. Then, Q is dense in $(\mathbb{P}, \mathcal{T}_f)$, while, by Corollary 2.4, $(Q, \mathcal{T}_f)$ is homeomorphic to $(\mathbb{Q}, \mathcal{T}_e)$ because $f|\mathcal{F}_2(Q) \in \mathcal{S}e\ell_2(Q, \mathcal{T}_e)$.

3. Selections generating new convergent sequences

In what follows, all spaces are assumed to be at least Hausdorff. Let $(X, \mathcal{T})$ be a space, and $p \in X$. A selection $f \in \mathcal{S}e\ell_2(X)$ is *p-maximal*, see [2, 4], if

$f(\{p,x\}) = p$ for every $x \in X$. Also, let us explicitly make the agreement that a family $\mathcal{B}$ of subsets of X is a $\mathcal{T}$*-base* in the point p if for every $\mathcal{T}$-neighbourhood U of p there exists $B \in \mathcal{B}$ such that $p \in \mathrm{Int}_{\mathcal{T}}(B) \subset B \subset U$. Here, $\mathrm{Int}_{\mathcal{T}}(B)$ means the *interior* of B with respect to the topology $\mathcal{T}$. Now, we shall say that a family $\mathcal{P}$ of subsets of X is a $\mathcal{T}$*-network* in p if for every $\mathcal{T}$-neighbourhood U of p there exists $P \in \mathcal{P}$ such that $p \in P \subset U$.

Following the proof of [4, Theorem 1.4], we have the following general construction of selections from given ones which provides, in particular, that $\mathcal{T}_f \neq \mathcal{T}$.

Theorem 3.1. *Let $(X, \mathcal{T})$ be a space, $p \in X$, and let $\{V_n : n < \omega\}$ be a decreasing family of $\mathcal{T}$-clopen sets such that $\{p\} = \bigcap\{V_n : n < \omega\}$. Also, suppose that $\mathcal{S}e\ell_2(X, \mathcal{T}) \neq \emptyset$. Then, $\mathcal{F}_2(X)$ has a p-maximal selection $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ such that $\{V_n : n < \omega\}$ is a $\mathcal{T}_f$-network in p. If, moreover, p is a non-isolated point, then $\{V_n : n < \omega\}$ is a $\mathcal{T}_f$-base in p.*

Proof. We can assume, without loss of generality, that $V_0 = X$, while, for convenience, we let $V_\omega = \{p\}$. Next, for every $x \in X$ define

$$n(x) = \max\{n \leq \omega : x \in V_n\}.$$

Since $\{p\} = \bigcap\{V_n : n < \omega\}$, we now have that $x = p$ if and only if $n(x) = \omega$. Keeping in mind this, take a selection $g \in \mathcal{S}e\ell_2(X, \mathcal{T})$, and then define another one $f \in \mathcal{S}e\ell_2(X)$ by letting for $x, y \in X$, with $n(x) \leq n(y)$, that

$$f(\{x,y\}) = \begin{cases} g(\{x,y\}) & \text{if} \quad n(x) = n(y), \\ y & \text{if} \quad n(x) < n(y). \end{cases}$$

Since $n(x) < \omega = n(p)$ for every $x \in X \setminus \{p\}$, we get a p-maximal selection for $\mathcal{F}_2(X)$. In order to observe that f is $\tau_{V(\mathcal{T})}$-continuous, take points $x, y \in X \setminus \{p\}$. If $n(x) = n(y)$, then f is $\tau_{V(\mathcal{T})}$-continuous at $\{x,y\}$ because $g|\mathcal{F}_2(V_{n(x)} \setminus V_{n(x)+1})$ is $\tau_{V(\mathcal{T})}$-continuous, while $V_{n(x)} \setminus V_{n(x)+1}$ is a $\mathcal{T}$-clopen subset. In case $n(x) \neq n(y)$, say $n(x) < n(y)$, by definition, we have that $f(\{t,z\}) = z$ for every $z \in V_{n(y)}$ and $t \in V_{n(x)} \setminus V_{n(y)}$. Hence, we get again that f is $\tau_{V(\mathcal{T})}$-continuous at $\{x,y\}$ because both $V_{n(x)} \setminus V_{n(y)}$ and $V_{n(y)}$ are $\mathcal{T}$-open, see, for instance, [3, Theorem 3.1]. On the other hand, f is $\tau_{V(\mathcal{T})}$-continuous at any singleton because it is a selection for $\mathcal{F}_2(X)$. Hence, it remains to show that f is $\tau_{V(\mathcal{T})}$-continuous at $\{x,p\}$ for every $x \in X \setminus \{p\}$. Take a point $x \in X \setminus \{p\}$, and a neighbourhood U of p, with $x \notin U$. Next, take an $n < \omega$ such that $n > n(x)$, and set $W = V_n \cap U$. According to our construction, we now have that $f(\{y,z\}) = z$ for every $z \in W$ and $y \in V_{n(x)} \setminus V_n$, so f is $\tau_{V(\mathcal{T})}$-continuous at $\{x,p\}$ as well. Thus, $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$.

Now, we are going to demonstrate that $\{V_n : n < \omega\}$ is a $\mathcal{T}_f$-network in p. To this end, let us observe that p is the $\preceq_f$-minimal element of X, hence the family of all $\preceq_f$-intervals

$$[p,x)_{\preceq_f} = \{y \in X : p \preceq_f y \prec_f x\} = (\leftarrow, x)_{\preceq_f}, \quad x \in X \setminus \{p\},$$

provides a subbase for the topology $\mathcal{T}_f$ in the point p. Take a non-empty finite subset $T \subset X \setminus \{p\}$. Then, there exists an $n < \omega$, with $V_n \cap T = \emptyset$. Hence,

$$V_n \subset \bigcap\{[p, x)_{\preceq_f} : x \in T\}.$$

Finally, suppose that p is a non-isolated point in X, and take $n < \omega$. Then, there is $m < \omega$ such that $V_n \setminus V_m \neq \emptyset$, so we can take a point $x \in V_n \setminus V_m$. However, this implies that $p \in V_m \subset [p, x)_{\preceq_f} \subset V_n$ because $n < m$, hence $p \in V_m \subset \mathrm{Int}_{\mathcal{T}_f}(V_n)$. That is, in this case, $\{V_n : n < \omega\}$ is a $\mathcal{T}_f$-base at p. □

The following is now an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let $(X, \mathcal{T})$ be a space, $p \in X$, $\{V_n : n < \omega\}$ be a decreasing family of $\mathcal{T}$-clopen sets such that $\{p\} = \bigcap\{V_n : n < \omega\}$, and let $F \subset X$ be a $\mathcal{T}$-closed set such that $p \notin F$ and $F \cap V_n \neq \emptyset$ for every $n < \omega$. Also, suppose that $\mathcal{S}e\ell_2(X, \mathcal{T}) \neq \emptyset$. Then, $\mathcal{F}_2(X)$ has a p-maximal selection $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ such that $\mathcal{T}_f \neq \mathcal{T}$.*

Proof. By Theorem 3.1, $\mathcal{F}_2(X)$ has a p-maximal selection $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ such that $\{V_n : n < \omega\}$ is a $\mathcal{T}_f$-network in p. Take a neighbourhood $U \in \mathcal{T}_f$ of p. Then, $U \cap F \neq \emptyset$ because $F \cap V_n \neq \emptyset$ for every $n < \omega$. Therefore, $p \in \overline{F}^{\preceq_f}$, while, by hypothesis, $p \notin F$. That is, $\mathcal{T}_f \neq \mathcal{T}$. □

On the other hand, we have the following general example.

Example 3.3. Let $(X, \mathcal{T})$ be a space which has an infinite discrete cover of $\mathcal{T}$-open subsets, and a point $p \in X$ which is a countable intersection of $\mathcal{T}$-clopen sets. Then, there exists a $\mathcal{T}$-closed subset $F \subset X$ and a decreasing family $\{V_n : n < \omega\}$ of $\mathcal{T}$-clopen subsets of X such that $p \notin F$, $\{p\} = \bigcap\{V_n : n < \omega\}$, and $F \cap V_n \neq \emptyset$ for every $n < \omega$.

Proof. By hypothesis, X has a discrete cover $\{S_n : n < \omega\} \subset \mathcal{T} \setminus \{\emptyset\}$, indexed one-to-one with the elements of ω, such that $p \in S_0$. On the other hand, there exists a decreasing family $\{T_n : n < \omega\}$ of $\mathcal{T}$-clopen subset such that $T_0 = S_0$ and $\{p\} = \bigcap\{T_n : n < \omega\}$. Next, for every $n < \omega$, set

$$V_n = T_n \cup \left(\bigcup\{S_k : k \geq n\}\right).$$

Then, $\{V_n : n < \omega\}$ is a decreasing family of $\mathcal{T}$-clopen sets, with $\{p\} = \bigcap\{V_n : n < \omega\}$, while $F = \bigcup\{S_{n+1} : n < \omega\}$ is a $\mathcal{T}$-closed subset of X such that $p \notin F$ and $F \cap V_n \neq \emptyset$ for every $n < \omega$. □

According to Corollary 3.2 and Example 3.3, we have also the following consequence.

Corollary 3.4. *If $(\mathbb{Q}, \mathcal{T}_e)$ is the space of the rational numbers, then there exists a selection $f \in \mathcal{S}e\ell_2(\mathbb{Q}, \mathcal{T}_e)$ such that $\mathcal{T}_f \neq \mathcal{T}_e$.*

Our next consequence provides a possible improvement of the construction stated in Theorem 3.1.

Corollary 3.5. *Let $(X, \mathcal{T})$ be a space, which has an infinite discrete $\mathcal{T}$-open cover, $p \in X$, and let $X_1, X_2 \subset X$ be $\mathcal{T}$-closed subsets such that $X_1 \cap X_2 = \{p\}$ and $X_1 \cup X_2 = X$. Suppose that $\mathcal{S}e\ell_2(X, \mathcal{T}) \neq \emptyset$, $X_1 \setminus \{p\}$ is connected in $(X, \mathcal{T})$, while p is a countable intersection of relatively $\mathcal{T}$-clopen subsets of X_2 as a subspace of $(X, \mathcal{T})$. Then, there exists $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ such that $\mathcal{T}_f \neq \mathcal{T}$.*

Proof. Let $\mathcal{T}_i$, $i = 1, 2$, be the relative topology on X_i as a subspace of $(X, \mathcal{T})$. Then, by hypothesis, the space $(X_2, \mathcal{T}_2)$ has an infinite discrete $\mathcal{T}_2$-open cover because $X_1 \setminus \{p\}$ is connected. Also, the point p is a countable intersection of $\mathcal{T}_2$-clopen subsets of X_2, while $\mathcal{S}e\ell_2(X_2, \mathcal{T}_2) \neq \emptyset$ because $\mathcal{S}e\ell_2(X, \mathcal{T}) \neq \emptyset$. Hence, by Corollary 3.2 and Example 3.3, there exists a p-maximal selection $f_2 \in \mathcal{S}e\ell_2(X_2, \mathcal{T}_2)$, with $(\mathcal{T}_2)_f \neq \mathcal{T}_2$. On the other hand, $\mathcal{F}_2(X_1)$ has a selection $f_1 \in \mathcal{S}e\ell_2(X_1, \mathcal{T}_1)$ such that $f_1(\{x, p\}) = x$ for every $x \in X_1$. Namely, if p is an isolated point of $(X_1, \mathcal{T}_1)$ this is obvious, note that $\mathcal{S}e\ell_2(X_1, \mathcal{T}_1) \neq \emptyset$. In case $(X_1, \mathcal{T}_1)$ is connected and $g \in \mathcal{S}e\ell_2(X_1, \mathcal{T}_1)$, then either $g(\{x, p\}) = p$ for every $x \in X_1$, or $g(\{x, p\}) = x$ for every $x \in X_1$. It is now clear that we may change the values of g if necessary to get the required selection f_1, see [3, Theorem 3.5]. Finally, we can define a selection $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ by letting $f|\mathcal{F}_2(X_i) = f_i$, $i = 1, 2$, and $f(\{x, y\}) = x$ if $x \in X_1$ and $y \in X_2$, see [2, Lemma 6.4]. This f is as required. □

Concerning the statements of Corollaries 3.2 and 3.5, let us explicitly mention that $\mathcal{T}_f = \mathcal{T}$ provided $(X, \mathcal{T})$ is compact and $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$, see [3, Proposition 4.1]. Also, $\mathcal{T}_f = \mathcal{T}$ provided $(X, \mathcal{T})$ is connected and locally connected, see [6, Lemma 11]. On the other hand, we have the following simple examples.

Example 3.6. There exists a locally connected space $(X, \mathcal{T})$ and $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$, with $\mathcal{T}_f \neq \mathcal{T}$.

Proof. Take $X = \{0\} \cup \{2^{-n} : n < \omega\}$, and $\mathcal{T}$ to be the discrete topology on X. Next, define $f : \mathcal{F}_2(X) \to X$ by $f(\{x, y\}) = \min\{x, y\}$, $x, y \in X$. Then, $\mathcal{T}_f$ is the Euclidean topology on X, hence $\mathcal{T}_f \neq \mathcal{T}$. □

Example 3.7. There exists a connected space $(X, \mathcal{T})$, and $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$, with $\mathcal{T}_f \neq \mathcal{T}$.

Proof. Let $X = [0, 1]$, and let $\mathcal{T}_e$ be the Euclidean topology on $[0, 1]$. Consider the topology $\mathcal{T}$ on X generated by $\mathcal{T}_e \cup \{X \setminus \{2^{-n} : n < \omega\}\}$. In fact, $\mathcal{T}$ is the modification of $\mathcal{T}$ in which the set $\{2^{-n} : n < \omega\}$ is closed. Now, just like before, define $f : \mathcal{F}_2(X) \to X$ by $f(\{x, y\}) = \min\{x, y\}$, $x, y \in X$. Then, $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ while $\mathcal{T}_f \neq \mathcal{T}$ because $\mathcal{T}_f = \mathcal{T}_e$. □

Let us stress the attention that the space $(X, \mathcal{T})$ of Example 3.7 is not regular, but a connected regular space with that property can be constructed as well. In fact, such examples are related to the following observation which is a simple consequence of [3, Theorem 3.1].

Proposition 3.8. *Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ be spaces, $f : (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ be a bijective continuous map, and let $g \in \mathcal{S}e\ell_2(X_2, \mathcal{T}_2)$. Define a map $h : \mathcal{F}_2(X_1) \to X_1$ by $h(\{x, y\}) = f^{-1}(g(\{f(x), f(y)\}))$, $x, y \in X_1$. Then, $h \in \mathcal{S}e\ell_2(X_1, \mathcal{T})$, where $\mathcal{T} = f^{-1}(\mathcal{T}_2)$, and, in particular, $h \in \mathcal{S}e\ell_2(X_1, \mathcal{T}_1)$.*

Proof. It is clear that h is a selection for $\mathcal{F}_2(X_1)$. Take two different points $x, y \in X_1$ such that $h(\{x, y\}) = x$. Then, by the definition of h, we have that $g(\{f(x), f(y)\}) = f(x)$. Hence, by [3, Theorem 3.1], there are disjoint sets $W_x, W_y \in \mathcal{T}_2$ such that $f(x) \in W_x$, $f(y) \in W_y$, and $g(\{t, z\}) = t$ for every $t \in W_x$ and $z \in W_y$. Let $V_x = f^{-1}(W_x)$, and $V_y = f^{-1}(W_y)$. Obviously, $V_x, V_y \in \mathcal{T} \subset \mathcal{T}_1$ are disjoint sets, $x \in V_x$, $y \in V_y$, and $h(\{t, z\}) = t$ for every $t \in V_x$ and $z \in V_y$. So, by [3, Theorem 3.1], $h \in \mathcal{S}e\ell_2(X_1, \mathcal{T})$. $\square$

Now, the reader may realize that our previous examples were based, in fact, on the following immediate consequence of Proposition 3.8.

Corollary 3.9. *Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ be spaces, $f : (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ be a bijective continuous map which is not a homeomorphism, and let $\mathcal{S}e\ell_2(X_2, \mathcal{T}_2) \neq \emptyset$. Then, there exists $h \in \mathcal{S}e\ell_2(X_1, \mathcal{T}_1)$, with $\mathcal{T}_h \neq \mathcal{T}_1$.*

Here is another example based on the same construction.

Example 3.10. There exists a connected, completely metrizable separable space $(X, \mathcal{T})$, and a selection $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ such that $\mathcal{T}_f \neq \mathcal{T}$.

Proof. Let $(X, \mathcal{T})$ be the subspace of the plane $\mathbb{R}^2$ defined by

$$X = \{(0, 0)\} \cup \{(x, \sin(1/x)) : 0 < x \leq 1\}.$$

Then, $(X, \mathcal{T})$ is connected, completely metrizable separable space which is not compact. Note that the projection $\pi : X \to [0, 1]$ onto the first factor is a bijective continuous map with respect to the Euclidean topology $\mathcal{T}_e$ on $[0, 1]$, but is not a homeomorphism because $([0, 1], \mathcal{T}_e)$ is compact. Since $\mathcal{S}e\ell_2([0, 1], \mathcal{T}_e) \neq \emptyset$, by Corollary 3.9, there exists $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ such that $\mathcal{T}_f \neq \mathcal{T}$. $\square$

Examples 3.7 and 3.10 are natural also from another point of view. Namely, concerning connected spaces, we have the following general result.

Theorem 3.11. *Let $(X, \mathcal{T})$ be a connected separable space, and let $\mathcal{T}_e$ be the Euclidean topology on $[0, 1]$. Then, $\mathcal{S}e\ell_2(X, \mathcal{T}) \neq \emptyset$ if and only if there exists an injective continuous map $h : (X, \mathcal{T}) \to ([0, 1], \mathcal{T}_e)$.*

Proof. We consider the non-trivial case when $|X| \geq 2$. Take an $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$. Then, by [5, Lemma 7.2], the f-order $\preceq_f$ on X is transitive, and, therefore, by Proposition 2.1, $(X, \mathcal{T}_f)$ is both connected and orderable. Then, let $(X^*, \mathcal{T}_f^*)$ be the orderable compactification of $(X, \mathcal{T}_f)$, see [1, 3.12.3(b)]. Note that $(X^*, \mathcal{T}_f^*)$ is also connected, hence it has exactly two continuous selection for $\mathcal{F}(X^*, \mathcal{T}_f^*)$ because it is compact and orderable. On the other hand, $(X^*, \mathcal{T}_f^*)$ is separable and infinite because so is $(X, \mathcal{T})$. Hence, by [6, Corollary 2], $(X^*, \mathcal{T}_f^*)$ is homeomorphic to $([0, 1], \mathcal{T}_e)$. The proof completes because one can take $h = 1_X : X \to X \subset X^*$ to be the identity map of X. □

On the other hand, let us observe that any one of the spaces in Examples 3.6, 3.7 and 3.10 contain, in fact, an infinite closed discrete subset. So, it seems natural to ask if a space $(X, \mathcal{T})$ contains an infinite discrete subset provided it has a selection $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$, with $\mathcal{T}_f \neq \mathcal{T}$. In general the answer is "No", here is an example based again on Corollary 3.9.

Example 3.12. There exists a sequentially compact first countable space $(X, \mathcal{T})$ and a selection $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ such that $\mathcal{T}_f \neq \mathcal{T}$.

Proof. Let $X = [0, \omega_1]$, where ω_1 is the first uncountable ordinal. Also, let $\mathcal{T}$ be the topology on X whose restriction on $[0, \omega_1)$ is the ordinal topology, while ω_1 is an isolated point in X. Then, $(X, \mathcal{T})$ is a sequentially compact, first countable, and non-compact space, see [1, Theorem 3.10.31]. On the other hand, if $\mathcal{T}_o$ is the ordinal topology on $[0, \omega_1]$, then $([0, \omega_1], \mathcal{T}_o)$ is a compact space, while the identity map $i : (X, \mathcal{T}) \to ([0, \omega_1], \mathcal{T}_o)$ is a continuous bijection. Hence, by Corollary 3.9, there exists $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$, with $\mathcal{T}_f \neq \mathcal{T}$, because $\mathcal{S}e\ell_2([0, \omega_1], \mathcal{T}_o) \neq \emptyset$. □

Motivated by these examples, we conclude this section with the following question.

▷ **Question 4.** Does there exists a space $(X, \mathcal{T})$, with $\mathcal{S}e\ell_2(X, \mathcal{T}) \neq \emptyset$, which is neither compact nor connected and locally connected but $\mathcal{T}_f = \mathcal{T}$ for every $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$?

4. The selection topology and continuity of selections

A common point of our previous examples is that the selection $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ is also continuous with respect to $\mathcal{T}_f$, i.e. $f \in \mathcal{S}e\ell_2(X, \mathcal{T}_f)$. Here is another construction, motivated by [3, Example 3.6], which provides, in particular, some counterexample about this as well.

Theorem 4.1. *Let $(X, \mathcal{T})$ be a space, $p, q \in X$ be two different points, and let $\{V_n : n < \omega\}$ and $\{W_n : n < \omega\}$ be decreasing families of $\mathcal{T}$-clopen subsets such that $\{p\} = \bigcap\{V_n : n < \omega\}$ and $\{q\} = \bigcap\{W_n : n < \omega\}$. Also, suppose that $\mathcal{S}e\ell_2(X, \mathcal{T}) \neq \emptyset$ and $F \subset W_0 \setminus \{q\}$ is a $\mathcal{T}$-clopen set. Then, there exists a selection*

$f \in \mathcal{S}e\ell_2(X, \mathcal{T})$ such that $\{V_n : n < \omega\}$ is a $\mathcal{T}_f$-network in p, while $\{W_n : n < \omega\}$ is a $\mathcal{T}_f$-network in q. If, moreover, $V_n \setminus V_{n+1} \neq \emptyset \neq F \cap (W_n \setminus W_{n+1})$ for every $n < \omega$, then $\{V_n : n < \omega\}$ is a $\mathcal{T}_f$-base in p, $\{W_n : n < \omega\}$ is $\mathcal{T}_f$-base in q, and $f \notin \mathcal{S}e\ell_2(X, \mathcal{T}_f)$.

Proof. The construction follows that one in [3, Example 3.6]. Namely, we can assume that $V_0 \cap W_0 = \emptyset$ and $V_0 \cup W_0 = X$. Also, for convenience, let $V_\omega = \{p\}$ and $W_\omega = \{q\}$. Next, for every $x \in X$ define $v(x) = \max\{n \leq \omega : x \in V_n\}$ if $x \in V_0$, and $w(x) = \max\{n \leq \omega : x \in W_n\}$ otherwise. Note that $v(x) = \omega$ iff $x = p$, and $w(x) = \omega$ iff $x = q$. Now, take a selection $g \in \mathcal{S}e\ell_2(X, \mathcal{T})$, $x, y \in X$, and, following Theorem 3.1, define a selection $f : \mathcal{F}_2(X) \to X$ as follows:

(1) $f(\{x, y\}) = g(\{x, y\})$ if either $v(x) = v(y)$ or $w(x) = w(y)$.

(2) $f(\{x, y\}) = x$ if either $v(x) > v(y)$ or $w(y) > w(x)$.

(3) $f(\{x, y\}) = x$ if $x \in V_0$, $y \in W_0$, and $w(y) \neq v(x) + 1$ or $y \notin F$.

(4) $f(\{x, y\}) = y$ if $x \in V_0$, $y \in F$, and $w(y) = v(x) + 1$.

First of all, let us check that $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$. To this end, let us observe that, just like in Theorem 3.1, $f|\mathcal{F}_2(V_0)$ is $\tau_{V(\mathcal{T})}$-continuous, and $f|\mathcal{F}_2(W_0)$ is $\tau_{V(\mathcal{T})}$-continuous. So, take points $x \in V_0$ and $y \in W_0$. We have the following possibilities. If $x \neq p$ and $y \neq q$, then, either $w(y) \neq v(x) + 1$ or $w(y) = v(x) + 1$.

In case $w(y) \neq v(x) + 1$, we have that $y \in W_{w(y)} \setminus W_{w(y)+1}$, $x \in V_{v(y)} \setminus V_{v(y)+1}$, and, by (3), $f(\{t, z\}) = t$ for every $t \in V_{v(x)} \setminus V_{v(x)+1}$ and $z \in W_{w(y)} \setminus W_{w(y)+1}$. Hence, by [3, Theorem 3.1], f is $\tau_{V(\mathcal{T})}$-continuous at $\{x, y\}$.

In case $w(y) = v(x) + 1$ and $y \notin F$, we have that $y \in W_{w(y)} \setminus \big(W_{w(y)+1} \cup F\big)$, $x \in V_{v(y)} \setminus V_{v(y)+1}$, while, by (3), $f(\{t, z\}) = t$ for every $t \in V_{v(x)} \setminus V_{v(x)+1}$ and $z \in W_{w(y)} \setminus \big(W_{w(y)+1} \cup F\big)$. So, just like before, by [3, Theorem 3.1], f is $\tau_{V(\mathcal{T})}$-continuous at $\{x, y\}$.

In case $w(y) = v(x) + 1$ and $y \in F$, we get that $y \in \big(W_{w(y)} \cap F\big) \setminus W_{w(y)+1}$ for every $x \in V_{v(y)} \setminus V_{v(y)+1}$, while, by (4), we have that $f(\{t, z\}) = z$ for every $t \in V_{v(x)} \setminus V_{v(x)+1}$ and $z \in \big(W_{w(y)} \cap F\big) \setminus W_{w(y)+1}$. Again, by [3, Theorem 3.1], f is $\tau_{V(\mathcal{T})}$-continuous at $\{x, y\}$.

Suppose now that $x = p$ and $y \neq q$. Then, we can take $n > w(y)$. So, by (3), $f(\{t, z\}) = t$ for every $t \in V_n$ and $z \in W_{w(y)} \setminus W_{w(y)+1}$, hence [3, Theorem 3.1] implies the $\tau_{V(\mathcal{T})}$-continuity of f at $\{x, y\}$. A quite similar case is when $x \neq p$ and $y = q$. Thus, we finally consider the case when $x = p$ and $y = q$. In this case, we rely on the properties of F. Namely, take an arbitrary $n < \omega$, and let $W = W_0 \setminus F$. Then, by (3), $f(\{t, z\}) = t$ for every $t \in V_n$ and $z \in W$, and, by [3, Theorem 3.1], f is $\tau_{V(\mathcal{T})}$-continuous at $\{x, y\}$. That is, we finally have that $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$.

To check that $\{V_n : n < \omega\}$ is a $\mathcal{T}_f$-network in p, note that p is the $\preceq_f$-minimal element of X, while q is the $\preceq_f$-maximal one. Take a point $y \in X \setminus \{p, q\}$. In case $y \in W_0$, by (3), $f(\{x, y\}) = x$ for every $x \in V_{w(y)}$. So, $p \in V_{w(y)} \subset (\leftarrow, y)_{\preceq_f} = [p, y)_{\preceq_f}$. If $y \in V_0$, then $v(y) < \omega$. So, by (2), $f(\{x, y\}) = x$ for every $x \in V_{v(y)+1}$, hence $V_{v(y)+1} \subset (\leftarrow, y)_{\preceq_f} = [p, y)_{\preceq_f}$ which completes the verification. In the same way we get that $\{W_n : n < \omega\}$ is a $\mathcal{T}_f$-network in q. Namely, take $x \in X \setminus \{p, q\}$. If $x \in W_0$, then $w(x) < \omega$. Hence, by (2), $f(\{x, y\}) = x$ for every $y \in W_{w(x)+1}$, so $W_{w(x)+1} \subset (x, \rightarrow)_{\preceq_f} = \{y \in X : x \prec_f y \preceq q\} = (x, q]_{\preceq_f}$. In case $x \in V_0$, by (3), we have $f(\{x, y\}) = x$ for every $y \in W_{v(x)+2}$, so $W_{v(x)+2} \subset (x, \rightarrow)_{\preceq_f} = (x, q]_{\preceq_f}$.

Finally, let us consider the case when $V_n \setminus V_n \neq \emptyset \neq F \cap (W_n \setminus W_{n+1})$ for every $n < \omega$. In this case, we first claim that $\{V_n : n < \omega\}$ is a $\mathcal{T}_f$-base in p. To this end, take an $n < \omega$, and $x_i \in V_{n+i} \setminus V_{n+i+1}$, $i = 0, 1$. Then, by (1), (2), (3) and (4),

$$(\leftarrow, x_i)_{\preceq_f} \subset V_{n+i} \cup ((W_{n+i+1} \cap F) \setminus W_{n+i+2}), \quad i = 0, 1.$$

Hence,

$$p \in (\leftarrow, x_0)_{\preceq_f} \cap (\leftarrow, x_1)_{\preceq_f} \subset V_n \cap V_{n+1} = V_{n+1} \subset V_n,$$

and therefore $p \in \mathrm{Int}_{\mathcal{T}_f}(V_n)$. The verification that $\{W_n : n < \omega\}$ is a $\mathcal{T}_f$-base in q is similar. Namely, take $y_i \in F \cap (W_{n+i} \setminus W_{n+i+1})$, $i = 1, 2$. Then, according to the definition of f,

$$(y_i, \rightarrow)_{\preceq_f} \subset (V_{n+i-1} \setminus V_{n+i}) \cup W_{n+i},$$

therefore

$$(y_1, \rightarrow)_{\preceq_f} \cap (y_2, \rightarrow)_{\preceq_f} \subset W_{n+1} \cap W_{n+2} \subset W_{n+1}.$$

Let us finally observe that this implies $f \notin \mathcal{S}e\ell_2(X, \mathcal{T}_f)$. Namely, suppose that f is $\tau_{V(\mathcal{T}_f)}$-continuous. Then, V_0 is a $\mathcal{T}_f$-neighbourhood of p, while $f(\{p, q\}) = p$. Since both $\{V_n : n < \omega\}$ and $\{W_n : n < \omega\}$ are decreasing, by [3, Theorem 3.1], there should be some $n < \omega$ such that $f(\{x, y\}) = x$ for every $x \in V_n$ and $y \in W_n$, while, by hypothesis, $(F \cap W_{n+1}) \setminus W_{n+2} \neq \emptyset \neq V_n \setminus V_{n+1}$. According to (4), this will imply that $f(\{x, y\}) = y$ for every $x \in V_n \setminus V_{n+1}$ and $y \in (F \cap W_{n+1}) \setminus W_{n+2} \subset W_n$. The contradiction so obtained completes the proof. □

By Theorem 4.1, we have the following simple consequence.

Corollary 4.2. *Let $(X, \mathcal{T})$ be a space, which has an infinite discrete $\mathcal{T}$-open cover, and at least two points which are countable intersections of $\mathcal{T}$-clopen subsets. Also, suppose $\mathcal{S}e\ell_2(X, \mathcal{T}) \neq \emptyset$. Then, there exists $f \in \mathcal{S}e\ell_2(X, \mathcal{T})$, with $f \notin \mathcal{S}e\ell_2(X, \mathcal{T}_f)$.*

Proof. Let $p, q \in X$ be as in the hypothesis, and let $\{S_n : n < \omega\} \subset \mathcal{T} \setminus \{\emptyset\}$ be a discrete cover of X, indexed one-to-one by the elements of ω, with $p \in S_0$ and $q \in S_1$. Since p is a countable intersection of $\mathcal{T}$-clopen subsets, there exists a decreasing family $\{T_n : n < \omega\}$ of $\mathcal{T}$-clopen subsets such that $T_0 = S_0$ and $\{p\} = \bigcap\{T_n : n < \omega\}$. In the same way, there exists a decreasing sequence $\{U_n : n < \omega\}$ of $\mathcal{T}$-clopen subsets such that $U_0 = S_1$ and $\{q\} = \bigcap\{U_n : n < \omega\}$. Then, for every $n < \omega$, set

$$V_n = T_n \cup \left(\bigcup\{S_{2k} : k \geq n\}\right) \quad \text{and} \quad W_n = U_n \cup \left(\bigcup\{S_{2k+1} : k \geq n\}\right).$$

Then, $\{V_n : n < \omega\}$ and $\{W_n : n < \omega\}$ are decreasing families of $\mathcal{T}$-clopen sets such that $\{p\} = \bigcap\{V_n : n < \omega\}$ and $\{q\} = \bigcap\{W_n : n < \omega\}$. On the other hand, $V_n \setminus V_{n+1} \neq \emptyset$ for every $n < \omega$, while $F = \bigcup\{S_{2k+1} : k \geq 1\} \subset W_0 \setminus \{q\}$ is a $\mathcal{T}$-clopen set such that $F \cap (W_n \setminus W_{n+1}) \neq \emptyset$ for every $n < \omega$. Hence, Theorem 4.1 completes the proof. □

According to Corollary 4.2 we now have the following two further consequences.

Corollary 4.3. *If $\mathcal{T}_e$ is the Euclidean topology on $\mathbb{Q}$, then there exists a selection $f \in \mathcal{S}e\ell_2(\mathbb{Q}, \mathcal{T}_e)$ such that $f \notin \mathcal{S}e\ell_2(\mathbb{Q}, \mathcal{T}_f)$.*

Corollary 4.4. *If $\mathcal{T}_e$ is the Euclidean topology on $\mathbb{P}$, then there exists a selection $f \in \mathcal{S}e\ell_2(\mathbb{P}, \mathcal{T}_e)$ such that $f \notin \mathcal{S}e\ell_2(\mathbb{P}, \mathcal{T}_f)$.*

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